Homework 1

1. Consider 3-dimensional manifold M. In analogy with $\mathbb{R}^3$ define on M the operators of divergence ($div\ \vec{v}$), curl ($rot\ \vec{v}$), and gradient ($grad\ f$) in terms of exterior derivative, d, its adjoint $d^\dagger$, etc. You will need to use the metric on M to relate vectors and one-forms, and $*$ to identify two-forms with one-forms. Can you comment on the relations $rot\ gradf = 0$ and $div\ rot\ \vec{v} = 0$? Are they true for the general 3-manifold?

2. Consider the complex projective space $\mathbb{CP}^n$ which is defined as collection of $n+1$ complex coordinates

$$(z_1, z_2, ..., z_{n+1}) \in \mathbb{C}^{n+1}$$

together with the identification

$$(z_1, z_2, ..., z_{n+1}) \sim (\lambda z_1, \lambda z_2, ..., \lambda z_{n+1}) \ , \qquad \lambda \in \mathbb{C} \ , \quad \lambda \neq 0 \ .$$

Please, show that $\mathbb{CP}^n$ is a complex manifold by giving the explicitly the holomorphic atlas. Explain why $\mathbb{CP}^n$ is a quotient $S^{2n+1}/U(1)$.

3. Consider a compact symplectic manifold M with a symplectic form ω (non-degenerate closed 2-form). Please show that ω cannot be exact. In other words, show that ω is non-trivial element in $H^2(M, \mathbb{R})$.

4. Take a Kähler manifold M and show that a Kähler form is co-closed.

5. Take $\mathbb{R}^2$ and remove the origin. Calculate the de Rham cohomology groups $H^\bullet(M, \mathbb{R})$ for this manifold $M = \mathbb{R}^2 - \{0\}$.

to be handed in before 5 p.m., March 15, 2010