Solution HW#2

Note: detailed steps of the solution which have been presented extensively during the Lectures may be left out.

We have to solve the wave equation in cartesian coordinates ignoring z. We solve the following equation:

$$\frac{\partial^2 u}{\partial t^2} = 4\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

with the specified boundary conditions using the method of separation of variables. Assuming that

$$u(x, y, t) = G(t)\phi(x, y)$$

we get the following pair of equations:

$$\begin{aligned} \frac{dG}{dt} &= -4\lambda G\,, \\ \nabla^2\phi + \lambda\phi &= 0\,, \end{aligned}$$

where the last line, the Helmholtz equation, is supplemented by boundary conditions. From the general properties of the Helmholtz problem we know that $\lambda \geq 0$. Separating variables further for the Helmholtz problem $\phi(x, y) = h(y)f(x)$ we get the following two equations with the boundary conditions:

$$\begin{aligned} &\frac{d^2 h}{dy^2} + \mu h = 0\,, \quad \frac{dh}{dy}(0) = 0\,, \quad h(2) = 0\,, \\ &\frac{d^2 f}{dx^2} + (\lambda - \mu)f = 0\,, \quad \frac{df}{dx}(0) = 0\,, \quad \frac{df}{dx}(3) = 0\,. \end{aligned}$$

Solving these equations and imposing boundary conditions we get

$$\begin{aligned} h(y) &= \cos\left(\frac{\pi}{2}\Big(\frac{1}{2} + m\Big)y\right), \quad \mu = \frac{\pi^2}{4}\Big(\frac{1}{2} + m\Big)^2, \quad m = 0, 1, 2, 3, \ldots \\ f(x) &= \cos\left(\frac{n\pi x}{3}\right), \quad \lambda - \mu = \left(\frac{n\pi}{3}\right)^2, \quad n = 0, 1, 2, 3, \ldots \end{aligned}$$

We solve the time-dependent equation as follows:

$$G(t) = A\cos(2\sqrt{\lambda}t) + B\sin(2\sqrt{\lambda}t)\,.$$

Thus the general solution of homogeneous part of the problem is

$$u(x, y, t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big[A_{nm}\cos(2\sqrt{\lambda_{nm}}t) + B_{nm}\sin(2\sqrt{\lambda_{nm}}t)\Big]\cos\left(\frac{n\pi x}{3}\right)\cos\left(\frac{\pi}{2}\left(\frac{1}{2} + m\right)y\right),$$

where

$$\lambda_{nm} = \left(\frac{n\pi}{3}\right)^2 + \frac{\pi^2}{4}\Big(\frac{1}{2} + m\Big)^2.$$

Imposing the initial condition we get $A_{nm} = 0$ and

$$2\sqrt{\lambda_{nm}} B_{nm} = \frac{\int_0^{x=3}\int_0^{y=2} g(x,y)\cos\left(\frac{n\pi x}{3}\right)\cos\left(\frac{\pi}{2}\left(\frac{1}{2}+m\right)y\right)\,dxdy}{\int_0^{x=3}\int_0^{y=2}\cos^2\left(\frac{n\pi x}{3}\right)\cos^2\left(\frac{\pi}{2}\left(\frac{1}{2}+m\right)y\right)\,dxdy}$$

Final answer:

$$u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin(2\sqrt{\lambda_{nm}}t)\cos\left(\frac{n\pi x}{3}\right)\cos\left(\frac{\pi}{2}\left(\frac{1}{2}+m\right)y\right),$$

where λ_{nm} is given above and B_{nm} is given by

$$B_{nm} = \frac{1}{2\sqrt{\lambda_{nm}}}\frac{\int_0^{x=3}\int_0^{y=2} g(x,y)\cos\left(\frac{n\pi x}{3}\right)\cos\left(\frac{\pi}{2}\left(\frac{1}{2}+m\right)y\right)\,dxdy}{\int_0^{x=3}\int_0^{y=2}\cos^2\left(\frac{n\pi x}{3}\right)\cos^2\left(\frac{\pi}{2}\left(\frac{1}{2}+m\right)y\right)\,dxdy}$$

The normalization of the eigenfunctions can also be worked out explicitly:

$$\int_0^{x=3}\int_0^{y=2}\cos^2\left(\frac{n\pi x}{3}\right)\cos^2\left(\frac{\pi}{2}\left(\frac{1}{2}+m\right)y\right)\,dxdy = \begin{cases} \frac{3}{2}, & n>0 \\ 3, & n=0 \end{cases}$$