Solution HW#5

We have to solve the non-homogenous wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + 5(e^{-5\pi t} - 1)\sin(4\pi x)\sin(2\pi y) + y$$

with one non-homogenous boundary condition, $u(x, 2, t) = x(1 - x)$. The Laplacian is given by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

We start by finding a function $r(x, y)$ that has the same non-homogeneous boundary value. For example,

$$r(x, y) = \frac{y}{2}x(1 - x)\,.$$

Writing $u = v + r$ and substituting it into the wave equation, we get

$$\frac{\partial^2 v}{\partial t^2} = \nabla^2 v + 5(e^{-5\pi t} - 1)\sin(4\pi x)\sin(2\pi y)$$

with $v = 0$ on all four boundaries $v(0, y, t) = v(1, y, t) = v(x, 0, t) = v(x, 2, t) = 0$.

We expand $v(x, y, t)$ in terms of the complete set of eigenfuctions to the homogeneous problem on the given rectangle

$$v(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}(t)\sin(n\pi x)\sin\Big(\frac{m\pi y}{2}\Big)\,.$$

Because the eigenfunctions satisfy the same boundary conditions as $v(x, y, t)$, we are allowed to differentiate term by term inside the sum. From the wave equation we obtain

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big(\frac{d^2 A_{mn}}{dt^2} - \lambda_{mn}A_{mn}\Big)\sin(n\pi x)\sin\Big(\frac{m\pi y}{2}\Big) = 5(e^{-5\pi t} - 1)\sin(4\pi x)\sin(2\pi y)\,,$$

where the eigenvalues are given by

$$\lambda_{mn} = n^2\pi^2 + \Big(\frac{m\pi}{2}\Big)^2\,.$$

Identifying the coefficients of the double-Fourier series on either side we get the equations

$$\frac{d^2 A_{mn}}{dt^2} + \lambda_{mn}A_{mn} = 0\,, \qquad n \neq 4 \text{ or } m \neq 4$$

$$\frac{d^2 A_{44}}{dt^2} + \lambda_{44}A_{44} = 5(e^{-5\pi t} - 1)\,,$$

where the last one is a non-homogenous ODE. This equation has a particular solution and a homogenous solution. Focusing on the particular solution we write down an ansatz that has the same type of objects as the right-hand side of the equation,

$$A_{44}(t) = a\,e^{-5\pi t} + b\,,$$

Plugging this into the non-homogenous ODE, we obtain

$$25\pi^2 a\, e^{-5\pi t} + \lambda_{44}(a\, e^{-5\pi t} + b) = 5(e^{-5\pi t} - 1)\,.$$

where $\lambda_{44} = 20\pi^2$. The solution is $a = \frac{1}{9\pi^2}$ and $b = -\frac{1}{4\pi^2}$. Combining the particular solution with the homogenous solutions, we have

$$\begin{aligned}
A_{mn}(t) &= A_{mn}\cos\left(\sqrt{\lambda_{mn}}t\right) + B_{mn}\sin\left(\sqrt{\lambda_{mn}}t\right), \qquad n \neq 4 \text{ or } m \neq 4\,,\\
A_{44}(t) &= A_{44}\cos\left(2\sqrt{5}\pi t\right) + B_{44}\sin\left(2\sqrt{5}\pi t\right) + \frac{1}{9\pi^2}\, e^{-5\pi t} - \frac{1}{4\pi^2}\,.
\end{aligned}$$

The initial condition given by

$$\frac{\partial u}{\partial t}(x, y, 0) = 0$$

implies that $A'_{mn}(0) = 0$. Thus we have the following equations for the B_{mn} coefficients:

$$\begin{aligned}
&B_{mn} = 0\,, \qquad n \neq 4 \text{ or } m \neq 4\,,\\
&2\sqrt{5}\pi B_{44} + \frac{-5\pi}{9\pi^2} = 0\,,
\end{aligned}$$

where the last equation gives the only nonzero B_{mn},

$$B_{44} = \frac{\sqrt{5}}{18\pi^2}\,.$$

The initial condition $u(x, y, 0) = f(x, y)$ gives the equation

$$f(x, y) - \frac{y}{2}x(1 - x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}(0)\sin(n\pi x)\sin\left(\frac{m\pi y}{2}\right),$$

where

$$\begin{aligned}
A_{mn}(0) &= A_{mn}\,, \qquad n \neq 4 \text{ or } m \neq 4\,,\\
A_{44}(0) &= A_{44} + \frac{1}{9\pi^2} - \frac{1}{4\pi^2}\,.
\end{aligned}$$

From the orthogonality of the eigenfunctions we also have that

$$A_{mn}(0) = 2\int_0^2\int_0^1 \left[f(x, y) - \frac{y}{2}x(1 - x)\right]\sin(n\pi x)\sin\left(\frac{m\pi y}{2}\right)dxdy\,.$$

For example, we now have the coefficient A_{44} expressed as

$$A_{44} = A_{44}(0) - \frac{1}{9\pi^2} + \frac{1}{4\pi^2}\,.$$

Final answer:

$$u(x,y,t) = \frac{y}{2}x(1-x) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}(t)\sin(n\pi x)\sin\Big(\frac{m\pi y}{2}\Big)\,.$$

where the functions $A_{mn}(t)$ are given by

$$A_{mn}(t) = A_{mn}(0)\cos\left(\sqrt{\lambda_{mn}}t\right), \qquad n \neq 4 \text{ or } m \neq 4\,,$$

$$A_{44}(t) = \Big(A_{44}(0) - \frac{1}{9\pi^2} + \frac{1}{4\pi^2}\Big)\cos\left(2\sqrt{5}\pi t\right) + \frac{\sqrt{5}}{18\pi^2}\sin\left(2\sqrt{5}\pi t\right) + \frac{1}{9\pi^2}\,e^{-5\pi t} - \frac{1}{4\pi^2}\,,$$

with eigenvalues

$$\lambda_{mn} = n^2\pi^2 + \Big(\frac{m\pi}{2}\Big)^2\,.$$

The coefficients $A_{mn}(0)$ are given by

$$A_{mn}(0) = 2\int_0^2\int_0^1 \Big[f(x,y) - \frac{y}{2}x(1-x)\Big]\sin(n\pi x)\sin\Big(\frac{m\pi y}{2}\Big)dxdy\,.$$