# Solution HW#1

We first solve the homogeneous part of the problem

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} - 4u\,, \quad u(0,t) = 0\,, \quad \frac{\partial u}{\partial x}(2\pi, t) = 0\,, \tag{1}$$

using the method of separation of variables. First we look for the solutions of the form $u(x,t) = \phi(x)G(t)$. By plugging $u(x,t) = \phi(x)G(t)$ into the above homogeneous PDE we obtain

$$\frac{1}{4G}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} - 1 = -\lambda\,, \tag{2}$$

where $\lambda$ is a separation constant. Thus we get two ODEs with homogeneous boundary conditions

$$\begin{aligned} &\frac{dG}{dt} = -4\lambda G\,, \\ &\frac{d^2\phi}{dx^2} = \Big(1-\lambda\Big)\phi\,, \quad \phi(0) = 0\,, \quad \frac{d\phi}{dx}(2\pi) = 0\,. \end{aligned} \tag{3}$$

Let us start from the second ODE with the boundary conditions. We look for the solutions of the form $\phi(x) = e^{\alpha x}$ and find that $\alpha = \pm i\sqrt{\lambda - 1}$. We have to consider the different behaviors of the solutions depending on the sign of the argument inside the square root:

$$\begin{aligned} &\lambda - 1 > 0\,, \quad \phi(x) = A\cos\Big(\sqrt{\lambda-1}\,x\Big) + B\sin\Big(\sqrt{\lambda-1}\,x\Big)\,, \\ &\lambda - 1 < 0\,, \quad \phi(x) = A\cosh\Big(\sqrt{1-\lambda}\,x\Big) + B\sinh\Big(\sqrt{1-\lambda}\,x\Big)\,, \end{aligned} \tag{4}$$

and finally

$$\lambda - 1 = 0\,, \quad \phi(x) = Ax + B. \tag{5}$$

The solutions for the last two cases, $\lambda - 1 < 0$ and $\lambda - 1 = 0$ cannot be made consistent with the boundary conditions, so we discard these.

Imposing the boundary conditions for the case $\lambda - 1 > 0$ we get

$$A = 0\,, \quad 2\pi\sqrt{\lambda - 1} = \frac{\pi}{2} + \pi n\,, \quad n = 0, 1, 2, 3, \ldots \tag{6}$$

so

$$\phi(x) = B\sin\Big(\Big(\frac{n}{2} + \frac{1}{4}\Big)x\Big)\,, \quad \lambda = \Big(\frac{n}{2} + \frac{1}{4}\Big)^2 + 1\,, \quad n = 0, 1, 2, 3, \ldots \tag{7}$$

The solution for the $t$-dependent equation is given by

$$G(t) = Ce^{-4\lambda t}. \tag{8}$$

The general solution of the homogeneous problem is

$$u(x,t) = \sum_{n=0}^{\infty} B_n \, e^{-\left(n+\frac{1}{2}\right)^2 t - 4t} \sin\Big(\Big(\frac{n}{2} + \frac{1}{4}\Big)x\Big)\,. \tag{9}$$

Imposing the initial condition we find $B_n$,

$$u(x,0) = \sum_{n=0}^{\infty} B_n \sin\Big(\Big(\frac{n}{2} + \frac{1}{4}\Big)x\Big) = f(x)\,,$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\Big(\Big(\frac{n}{2} + \frac{1}{4}\Big)x\Big)dx\,. \tag{10}$$