Solution HW#4

We have to solve Laplace's equation in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial u}{\partial\theta}\Big) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0\,.$$

First we will find the general solution of the homogeneous part of the problem. Assuming that

$$u(r,\theta,\phi) = f(r)h(\theta)g(\phi)\,,$$

we get the following equations with homogenous boundary conditions:

$$\begin{aligned}
&\frac{d^2 g}{d\phi^2} = -\mu g\,, \quad g(0) = 0\,, \quad g(\pi) = 0\,, \\
&\frac{d}{d\theta}\Big(\sin\theta\frac{dh}{d\theta}\Big) + \Big(\lambda\sin\theta - \frac{\mu}{\sin\theta}\Big)h = 0\,, \quad |h(0)| < \infty\,, \quad |h(\pi)| < \infty\,, \\
&r^2\frac{d^2 f}{dr^2} + 2r\frac{df}{dr} - \lambda f = 0\,, \quad |f(0)| < \infty\,,
\end{aligned}$$

where the singularity conditions are needed since $\theta = 0, \pi$ and $r = 0$ are points on the sphere where we do not expect singular behavior.

We start solving from the first equation. The first equation has a solution only for $\mu \geq 0$

$$g(\phi) = A\cos(m\phi) + B\sin(m\phi)\,, \quad \mu = m^2\,, \quad m = 0, 1, 2, 3, \dots\,,$$

where $m = 0$ exist only for the cosine. The second equation now has the form

$$\frac{d}{d\theta}\Big(\sin\theta\frac{dh}{d\theta}\Big) + \Big(\lambda\sin\theta - \frac{m^2}{\sin\theta}\Big)h = 0\,, \quad |h(0)| < \infty\,, \quad |h(\pi)| < \infty\,.$$

which becomes

$$\frac{d}{dx}\Big((1-x^2)\frac{dh}{dx}\Big) + \Big(\lambda - \frac{m^2}{1-x^2}\Big)h = 0\,, \quad |h(1)| < \infty\,, \quad |h(-1)| < \infty\,.$$

after a change of variables $x = \cos\theta$. In order to have a finite solution at $x = \pm 1$ we have to require $\lambda = l(l+1)$, with l being an integer such that $l \geq m$, giving associated Legendre functions

$$h(\theta) = CP_l^m(\cos\theta)\,.$$

For f we have an equidimensional equation and thus we can try the ansatz $f(r) = r^\alpha$, giving the equation

$$\alpha(\alpha-1) - 2\alpha + l(l+1) = 0$$

which has the two solutions $\alpha = l$ and $\alpha = -l-1$. We conclude that

$$f(r) = Dr^l + Er^{-l-1}\,,$$

and due to the singularity condition we keep only the solution that is finite at zero.

Thus the general solution of the homogeneous PDE with homogeneous BC is

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} A_{lm}\, r^l\, P_l^m(\cos\theta)\cos(m\phi) + \sum_{l=1}^{\infty}\sum_{m=1}^{l} B_{lm}\, r^l\, P_l^m(\cos\theta)\sin(m\phi)\,.$$

Using the orthogonality of the generalized Fourier series, the coefficients can be computed from the remaining boundary condition $u(6,\theta,\phi) = \sin^2\!\theta\cos(2\phi) + \sin\theta\cos\phi \equiv F(\theta,\phi)$

$$A_{lm} = \frac{1}{6^l}\frac{\int_{-\pi}^{\pi}\int_0^{\pi} F(\theta,\phi)\cos(m\phi)P_l^m(\cos\theta)\, d(\cos\theta)d\phi}{\int_{-\pi}^{\pi}\int_0^{\pi}\cos^2(m\phi)(P_l^m(\cos\theta))^2\, d(\cos\theta)d\phi}\,,$$
$$B_{lm} = \frac{1}{6^l}\frac{\int_{-\pi}^{\pi}\int_0^{\pi} F(\theta,\phi)\sin(m\phi)P_l^m(\cos\theta)\, d(\cos\theta)d\phi}{\int_{-\pi}^{\pi}\int_0^{\pi}\sin^2(m\phi)(P_l^m(\cos\theta))^2\, d(\cos\theta)d\phi}\,.$$

However, given the simplicity of the functions appearing in the non-homogeneous boundary condition there is a simpler approach. The boundary condition function can be directly expressed in terms of the eigenfunctions on the sphere

$$u(6,\theta,\phi) = \sin^2\!\theta\cos(2\phi) + \sin\theta\cos\phi = \frac{1}{3}P_2^2(\cos\theta)\cos(2\phi) + P_1^1(\cos\theta)\cos\phi\,.$$

Thus by orthogonality of the eigenfunctions, only the A_{lm} coefficients with $l = m = 2$ and $l = m = 1$ are non-vanishing. The general solution then collapses to

$$u(r,\theta,\phi) = A_{11}\, r\, P_1^1(\cos\theta)\cos\phi + A_{22}\, r^2 P_2^2(\cos\theta)\cos(2\phi)\,.$$

Now we can term-by-term impose the non-homogeneous boundary condition, giving

$$A_{11} = \frac{1}{6}\,, \quad A_{22} = \frac{1}{3\times 6^2}\,,$$

thus we have the final solution

$$\begin{aligned} u(r,\theta,\phi) &= \frac{r}{6}\, P_1^1(\cos\theta)\cos\phi + \frac{1}{3}\Big(\frac{r}{6}\Big)^2 P_2^2(\cos\theta)\cos(2\phi) \\ &= \frac{r}{6}\sin\theta\cos\phi + \Big(\frac{r}{6}\Big)^2\sin^2\!\theta\cos(2\phi)\,. \end{aligned}$$