Solution HW#6

This problem is best solved by splitting it into two simpler problems $u(x,t) = u_1(x,t) + u_2(x,t)$, where

$$\begin{aligned} &\frac{\partial^2 u_1}{\partial t^2} = 4\frac{\partial^2 u_1}{\partial x^2}\,, \\ &u_1(0,t) = 0\,, \\ &u_1(x,0) = f(x)\,, \quad \frac{\partial u_1}{\partial t}(x,0) = 0\,, \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^2 u_2}{\partial t^2} = 4\frac{\partial^2 u_2}{\partial x^2}\,, \\ &u_2(0,t) = g(t)\,, \\ &u_2(x,0) = 0\,, \quad \frac{\partial u_2}{\partial t}(x,0) = 0\,. \end{aligned}$$

The solution for u_1 is found by considering the solutions to the PDE that satisfy the homogenous BC and IC, and superposing them,

$$u_1(x,t) = \int_0^\infty A(\omega)\sin(\omega x)\cos(2\omega t)d\omega\,.$$

Imposing the non-homogenous initial condition $u_1(x,0) = f(x)$ gives the Fourier transform

$$f(x) = \int_0^\infty A(\omega)\sin(\omega x)d\omega\,,$$

which can be inverted

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)dx\,.$$

Finding the solution for u_2 is more complicated since the equations are not on a familiar form. In particular, the initial conditions of u_2 implies that both the position and the velocity of the physical object should vanish at $t = 0$. Naively this seems to imply that u_2 vanishes identically, however, this can only be true in a neighborhood of $t \approx 0$, and not for large (positive) values of t.

Being very careful, we proceed by writing down the most general solution to the PDE (ignoring the boundary and initial conditions),

$$u_2(x,t) = \int_{-\infty}^\infty C_1(\omega)\,e^{i\omega x}\,e^{i2\omega t}d\omega + \int_{-\infty}^\infty C_2(\omega)\,e^{-i\omega x}\,e^{i2\omega t}d\omega\,.$$

Imposing the non-homogenous boundary condition we have,

$$g(t) = \int_{-\infty}^{\infty} C_1(\omega)\, e^{i2\omega t} d\omega + \int_{-\infty}^{\infty} C_2(\omega)\, e^{i2\omega t}\, d\omega\,,$$

which is a sum of Fourier transforms of two functions (albeit with non-standard normalization in the exponent). Let us call the first integral $g_1(t)$ and the second one $g_2(t)$, such that $g(t) = g_1(t) + g_2(t)$, and then proceed to look at the initial conditions.

Analyzing the $u_2(x, 0) = 0$ initial condition we get that

$$0 = \int_{-\infty}^{\infty} C_1(\omega)\, e^{i\omega x} d\omega + \int_{-\infty}^{\infty} C_2(\omega)\, e^{-i\omega x} d\omega = g_1(x/2) + g_2(-x/2)\,,$$

where we have identified that that the x-dependent Fourier transforms are identical to the above t-dependent Fourier transforms after letting $t \to \pm x/2$. We conclude that $g_2(-x/2) = -g_1(x/2)$ (for positive x).

Similarly, analyzing the $\frac{\partial u_2}{\partial t}(x, 0) = 0$ initial condition we get that

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} C_1(\omega)\, i2\omega\, e^{i\omega x} d\omega + \int_{-\infty}^{\infty} C_2(\omega)\, i2\omega\, e^{-i\omega x} d\omega \\ &= 2\frac{d}{dx}\int_{-\infty}^{\infty} C_1(\omega)\, e^{i\omega x} d\omega - 2\frac{d}{dx}\int_{-\infty}^{\infty} C_2(\omega)\, e^{-i\omega x} d\omega \\ &= 2\frac{d}{dx}\Big(g_1(x/2) - g_2(-x/2)\Big) = 4\frac{d}{dx}\Big(g_1(x/2)\Big)\,, \end{aligned}$$

Hence $g_1(t) = g_2(-t) = c$ is a constant. We can find this constant using that $g(0) = g_1(0) + g_2(0) = 2c = 0$. Thus $g_1(t) = 0$ identically, and similarly $g_2(-t) = 0$; although, the latter need not be zero for positive arguments. Indeed, we can only make sense of these equations if we have

$$g_2(t) = \begin{cases} g(t)\,, & t \geq 0 \\ 0\,, & t < 0 \end{cases}\,.$$

Now we can invert the Fourier transform that the boundary condition imposed,

$$C_1(\omega) = 0\,, \qquad C_2(\omega) = \frac{1}{\pi}\int_0^{\infty} g(t)\, e^{-i2\omega t}\, dt\,,$$

where the integration range is $0 \leq t < \infty$ since negative time gives no contribution. (The normalization is here obtained through $\frac{1}{2\pi} d(2t) = \frac{1}{\pi} dt$.)

Thus we have the final solution

$$u(x, t) = \int_0^{\infty} A(\omega) \sin(\omega x) \cos(2\omega t) d\omega + \int_{-\infty}^{\infty} C_2(\omega)\, e^{-i\omega x}\, e^{i2\omega t} d\omega\,,$$

with the above expressions for $A(\omega)$ and $C_2(\omega)$.

It is interesting to note that we can further simplify the solution. By using the trigonometric identity $\sin(\omega x)\cos(2\omega t) = \frac{1}{2}\sin(\omega(x+2t)) + \frac{1}{2}\sin(\omega(x-2t))$ we have

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\int_0^\infty A(\omega)\sin\big(\omega(x+2t)\big)d\omega + \frac{1}{2}\int_0^\infty A(\omega)\sin\big(\omega(x-2t)\big)d\omega \\
&\quad + \int_{-\infty}^\infty C_2(\omega)\, e^{i2\omega(t-x/2)}d\omega \\
&= \frac{1}{2}f(x+2t) + \frac{1}{2}f(x-2t) + g_2(t-x/2)
\end{aligned}
$$

where we have identified the Fourier transforms of the functions $f(x)$ and $g_2(t)$ with shifted arguments. Note that, in order to be precise, we need to define $f(x)$ for negative arguments. Since $\sin(\omega x)$ is an odd function, if follows that the Fourier sine transform imposes that $f(-x) = -f(x)$.

Given the simplicity of the final solution, we should have realized that a simpler approach to solving this problem would be to look for solutions on the form

$$
u_{1,2}(x,t) = F(x+2t) + G(x-2t)\,,
$$

which satisfy the BC and IC. Thinking about the physical problem, it should be clear that $u_1(x,t)$ consists of waves traveling both left and right $u_1(x,t) = \frac{1}{2}f(x+2t) + \frac{1}{2}f(x-2t)$, sourced by the function $f(x)$ at $t = 0$. And $u_2(x,t)$ consist of waves only traveling to the right $u_2(x,t) = g_2(t-x/2)$, because these waves are sourced at the left boundary ($x = 0$) by the function $g(t)$, and then travel towards the right ($x > 0$) forward in time ($t > 0$).