Solutions for the final exam, 30-05-2016 FMM F

Problem 1

We have to solve

$$2\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} - u\,.$$

Assuming that $u(x,t) = G(t)\phi(x)$ we get

$$\frac{1}{4G}\Big(2\frac{d^2G}{dt^2} + G\Big) = \frac{1}{\phi}\frac{d^2\phi}{dx^2} \equiv -\lambda\,,$$

which implies the following ODEs

$$\frac{d^2G}{dt^2} + \Big(2\lambda + \frac{1}{2}\Big)G = 0\,,$$
$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0\,,$$

with the last equation supplemented by the boundary conditions

$$\frac{d\phi}{dx}(0) = 0\,, \qquad \phi(2) = 0\,.$$

For $\lambda > 0$ we have the solution

$$\phi(x) = A\cos\left(\sqrt{\lambda}\,x\right) + B\sin\left(\sqrt{\lambda}\,x\right)\,,$$

where the boundary conditions impose that

$$B = 0\,, \qquad \sqrt{\lambda} = \frac{\pi}{2}\Big(n + \frac{1}{2}\Big)\,, \qquad n = 0, 1, 2, \ldots$$

For $\lambda = 0$ we have the solution $\phi(x) = a + bx$, but the BC impose $a = b = 0$. For $\lambda < 0$ the solution is in terms of hyperbolic functions that cannot be made consistent with the BC.

The G-dependent equation is solved as follows

$$G(t) = C\cos\Big(t\sqrt{2\lambda + \frac{1}{2}}\Big) + D\sin\Big(t\sqrt{2\lambda + \frac{1}{2}}\Big)\,.$$

The homogeneous initial condition impose that $G(0) = 0$, implying that $C = 0$.

Combing everything together we get the following general solution

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\Big(\frac{\pi x}{4}(2n+1)\Big)\sin\Big(t\sqrt{\frac{\pi^2}{8}(2n+1)^2 + \frac{1}{2}}\Big)\,.$$

Finally, we impose the last non-homogeneous initial condition: $\frac{\partial u}{\partial t}(x,0) = g(x)$. Using the orthogonality of cosine functions on the interval $[0,2]$, we arrive at

$$A_n = \frac{1}{\sqrt{\frac{\pi^2}{8}(2n+1)^2 + \frac{1}{2}}} \int_0^2 g(x) \cos\left(\frac{\pi x}{4}(2n+1)\right) dx \,.$$

Problem 2

We have to solve the Laplace equation $\nabla^2 u = 0$ in cylindrical coordinates,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \,.$$

Assuming that $u(\rho, \phi, z) = F(\rho)h(\phi)G(z)$ we get the following set of ODEs

$$\begin{aligned}
&\frac{d^2 h}{d\phi^2} = -\mu h \,, \quad h(-\pi) = h(\pi) \,, \quad \frac{dh}{d\phi}(-\pi) = \frac{dh}{d\phi}(\pi) \,, \\
&\rho^2 \frac{d^2 F}{d\rho^2} + \rho \frac{dF}{d\rho} + (\lambda \rho^2 - \mu) F = 0 \,, \quad F(5) = 0 \,, \quad |F(0)| < \infty \,, \\
&\frac{d^2 G}{dz^2} = \lambda G \,,
\end{aligned}$$

with homogeneous BCs that either are given or obtained from physical considerations.

The first ODE + BC has a solution only for $\mu \geq 0$,

$$h(\phi) = A\cos(m\phi) + B\sin(m\phi) \,, \quad \mu = m^2 \,, \quad m = 0, 1, 2, \ldots$$

and the second ODE + BC has a solution only for $\lambda > 0$,

$$F(\rho) = C J_m(\sqrt{\lambda}\rho) \,, \quad \lambda = \left(\frac{Z_{mn}}{5}\right)^2 \,,$$

where Z_{mn} are the zeros of the Bessel function J_m; that is, the Z_{mn} are defined to satisfy

$$J_m(Z_{mn}) = 0 \,, \qquad n = 1, 2, 3, \ldots$$

The solution for G is

$$G(t) = D\sinh(\sqrt{\lambda} z) + E\sinh\left(\sqrt{\lambda}(2 - z)\right) \,,$$

where we have anticipated that it is useful to work with eigenfunctions that vanish on one of the boundaries boundaries, $z = 0, 2$ respectively.

Assembling the general solution for the homogeneous part of the problem, we get

$$
\begin{aligned}
u(\rho,\phi,z) = &\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big(A'_{mn}\cos(m\phi)+B'_{mn}\sin(m\phi)\Big)J_m\Big(\frac{Z_{mn}\rho}{5}\Big)\sinh\Big(\frac{Z_{mn}z}{5}\Big)\\
&+\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big(A''_{mn}\cos(m\phi)+B''_{mn}\sin(m\phi)\Big)J_m\Big(\frac{Z_{mn}\rho}{5}\Big)\sinh\Big(\frac{Z_{mn}}{5}(2-z)\Big)\,,
\end{aligned}
$$

where we have combined the unknown coefficients into more convenient objects.

Imposing the non-homogeneous initial conditions $u(\rho,\phi,0) = f(\rho,\phi)$ and $u(\rho,\phi,2) = g(\rho,\phi)$ we get two decoupled equations

$$
\begin{aligned}
f(\rho,\phi) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big(A''_{mn}\cos(m\phi)+B''_{mn}\sin(m\phi)\Big)J_m\Big(\frac{Z_{mn}\rho}{5}\Big)\sinh\Big(\frac{2Z_{mn}}{5}\Big)\,,\\
g(\rho,\phi) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Big(A'_{mn}\cos(m\phi)+B'_{mn}\sin(m\phi)\Big)J_m\Big(\frac{Z_{mn}\rho}{5}\Big)\sinh\Big(\frac{2Z_{mn}}{5}\Big)\,.
\end{aligned}
$$

The unknowns are obtained using orthogonality of Bessel and cos/sin functions; they are given by

$$
\begin{aligned}
A'_{mn} &= \frac{1}{\sinh\big(\frac{2Z_{mn}}{5}\big)}\frac{\int_{-\pi}^{\pi}\int_0^5 g(\rho,\phi)\,J_m\big(\frac{Z_{mn}\rho}{5}\big)\cos(m\phi)\rho\,d\rho\,d\phi}{\int_{-\pi}^{\pi}\int_0^5 J_m^2\big(\frac{Z_{mn}\rho}{5}\big)\cos^2(m\phi)\rho\,d\rho\,d\phi}\,,\\
B'_{mn} &= \frac{1}{\sinh\big(\frac{2Z_{mn}}{5}\big)}\frac{\int_{-\pi}^{\pi}\int_0^5 g(\rho,\phi)\,J_m\big(\frac{Z_{mn}\rho}{5}\big)\sin(m\phi)\rho\,d\rho\,d\phi}{\int_{-\pi}^{\pi}\int_0^5 J_m^2\big(\frac{Z_{mn}\rho}{5}\big)\sin^2(m\phi)\rho\,d\rho\,d\phi}\,, \qquad (m>0)
\end{aligned}
$$

with identical expressions for $(A' \to A'', g \to f)$ and $(B' \to B'', g \to f)$.

Problem 3

We have to solve

$$
\frac{1}{5}\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial u}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\,.
$$

The problem is solved by separation of variables, $u(r,\theta,\phi,t) = F(r)w(\theta)h(\phi)G(t)$, giving the following set of ODEs

$$
\begin{aligned}
&\frac{d^2h}{d\phi^2} = -m^2 g\,, \qquad h(-\pi) = h(\pi)\,, \quad \frac{dh}{d\phi}(-\pi) = \frac{dh}{d\phi}(\pi)\,,\\
&\frac{d}{d\theta}\Big(\sin\theta\,\frac{dw}{d\theta}\Big) + \Big(\mu\sin\theta - \frac{m^2}{\sin\theta}\Big)w = 0\,, \quad |w(0)| < \infty\,, \quad |w(\pi)| < \infty\,,\\
&r^2\frac{d^2F}{dr^2} + 2r\frac{dF}{dr} + (\lambda r^2 - \mu)F = 0\,, \quad F(3) = 0\,, \quad |F(0)| < \infty\,.\\
&\frac{dG}{dt} = -5\lambda G\,.
\end{aligned}
$$

The first equation we already solved in Problem 2,

$$h(\phi) = A\cos(m\phi) + B\sin(m\phi)\,, \quad m = 0, 1, 2, \ldots$$

For the second equation, in order to have a finite solution everywhere, we have to require $\mu = l(l+1)$, $l = 0, 1, 2, \ldots$; giving associated Legendre functions,

$$w(\theta) = C P_l^m(\cos\theta)\,,$$

where $l \geq m$.

Given that $\mu = l(l+1)$ the equation for F is now the PDE for spherical Bessel functions. Assuming regularity at the origin, and $F(3) = 0$, the solution is

$$F(r) = \frac{D}{\sqrt{r}}\,J_{l+\frac{1}{2}}\Big(\frac{z_{ln}r}{3}\Big)\,, \quad \lambda = \Big(\frac{z_{ln}}{3}\Big)^2\,,$$

where z_{ln} is the n'th zero; i.e. $J_{l+\frac{1}{2}}(z_{ln}) = 0$. Finally, the time-dependent equation is easily solved $G(t) = e^{-5\lambda t}$.

The general solution to the heat equation inside of a sphere is then

$$u(r,\phi,\theta,t) = \sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{1}{\sqrt{r}}\,J_{l+\frac{1}{2}}\Big(\frac{z_{ln}r}{3}\Big)\,P_l^m(\cos\theta)\,\big(A_{nlm}\cos(m\phi) + B_{nlm}\sin(m\phi)\big)e^{-\frac{5}{9}z_{ln}^2 t}\,.$$

Imposing the initial condition

$$u(r,\theta,\phi,0) = f(r)\sin^2\theta\cos(2\phi) + g(r)\,,$$

we see that $B_{nlm} = 0$ and furthermore the solution has to be a sum over two types of contributions: $m = 2$ and $l = m = 0$. The general solution then simplifies to

$$
\begin{aligned}
u(r,\phi,\theta,t) = &\sum_{n=1}^{\infty}\sum_{l=2}^{\infty} A_{nl2}\frac{1}{\sqrt{r}}\,J_{l+\frac{1}{2}}\Big(\frac{z_{ln}r}{3}\Big)\,P_l^2(\cos\theta)\cos(2\phi)e^{-\frac{5}{9}z_{ln}^2 t}\\
&+ \sum_{n=1}^{\infty} A_{n00}\frac{1}{\sqrt{r}}\,J_{\frac{1}{2}}\Big(\frac{z_{0n}r}{3}\Big)\,e^{-\frac{5}{9}z_{0n}^2 t}\,.
\end{aligned}
$$

and using the orthogonality of spherical Bessel functions and similarly orthogonality of associated Legendre functions we can fix the remaining unknown coefficients,

$$A_{n00} = \frac{\int_0^3 g(r) \frac{1}{\sqrt{r}} J_{\frac{1}{2}}\left(\frac{z_{0n} r}{3}\right) r^2 dr}{\int_0^3 J^2_{\frac{1}{2}}\left(\frac{z_{0n} r}{3}\right) r dr},$$

$$A_{nl2} = \frac{\int_0^3 f(r) \frac{1}{\sqrt{r}} J_{l+\frac{1}{2}}\left(\frac{z_{ln} r}{3}\right) r^2 dr}{\int_0^3 J^2_{l+\frac{1}{2}}\left(\frac{z_{ln} r}{3}\right) r dr} \times \frac{\int_0^\pi \sin^2\theta \, P_l^2(\cos\theta) \, d(\cos\theta)}{\int_0^\pi \left(P_l^2(\cos\theta)\right)^2 d(\cos\theta)}.$$

We can further simplify this by realizing that for $x = \cos\theta$ we have

$$\sin^2\theta = 1 - x^2 = (1 - x^2) \frac{1}{3} \frac{d^2}{dx^2} P_2(x) = \frac{1}{3} P_2^2(x),$$

where $P_2(x) = (3x^2 - 1)/2$. Thus we conclude that $A_{nl2} = 0$, unless $l = 2$, in which case

$$A_{n22} = \frac{1}{3} \frac{\int_0^3 f(r) \frac{1}{\sqrt{r}} J_{\frac{5}{2}}\left(\frac{z_{2n} r}{3}\right) r^2 dr}{\int_0^3 J^2_{\frac{5}{2}}\left(\frac{z_{2n} r}{3}\right) r dr}.$$

And the final answer is:

$$u(r, \phi, \theta, t) = \sum_{n=1}^{\infty} A_{n22} \frac{3}{\sqrt{r}} J_{\frac{5}{2}}\left(\frac{z_{2n} r}{3}\right) \sin^2\theta \cos(2\phi) e^{-\frac{5}{9} z_{2n}^2 t} + \sum_{n=1}^{\infty} A_{n00} \frac{1}{\sqrt{r}} J_{\frac{1}{2}}\left(\frac{z_{0n} r}{3}\right) e^{-\frac{5}{9} z_{0n}^2 t}.$$

Problem 4

We first make the boundary conditions homogeneous by $v(x, y, t) = u(x, y, t) - yx(1 - x)$. After substituting this into the PDE the $2y$-term cancels out and we obtain

$$\frac{\partial^2 v}{\partial t^2} = \nabla^2 v + 5e^{-5\pi t} \sin(4\pi x) \sin(3\pi y),$$

now with $v = 0$ on all four boundaries. We use the method of eigenfunction expansion, the solution should admit the form

$$v(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm}(t) \sin(n\pi x) \sin(m\pi x),$$

which satisfies the homogeneous boundary conditions automatically. After substituting this solution into the PDE and identifying the Fourier coefficients on the two sides, we get the following set of ODEs

$$\begin{aligned}\frac{d^2a_{nm}}{dt^2} &= -\pi^2(n^2+m^2)a_{nm}\,, \qquad n \neq 4 \text{ or } m \neq 3\\ \frac{d^2a_{43}}{dt^2} &= -25\pi^2 a_{43} + 5e^{-5\pi t}\,,\end{aligned}$$

together with homogenous initial condition $\frac{da_{nm}}{dt}(0) = 0$. The first line has the solution

$$a_{nm}(t) = a_{nm}(0)\cos\left(\pi t\sqrt{n^2+m^2}\right), \quad n \neq 4 \text{ or } m \neq 3,,$$

The second line equation has similar homogenous solution, $a_{43}(t) \sim \cos(5\pi t)$ or $a_{43}(t) \sim \sin(5\pi t)$, and the complete solution can be found using the ansatz

$$a_{43}(t) = A\cos(5\pi t) + B\sin(5\pi t) + Ce^{-5\pi t}\,,$$

which gives the following equation

$$25\pi^2 Ce^{-5\pi t} = -25\pi^2 Ce^{-5\pi t} + 5e^{-5\pi t}\,,$$

solved by $C = \frac{1}{10\pi^2}$. Imposing the initial conditions of the ODE, we find that $B = C$ and $A = a_{43}(0) - C$,

$$a_{43}(t) = \Big(a_{43}(0) - \frac{1}{10\pi^2}\Big)\cos(5\pi t) + \frac{1}{10\pi^2}\Big(\sin(5\pi t) + e^{-5\pi t}\Big)\,.$$

Thus the solution to the original PDE is given by

$$\begin{aligned}u(x,y,t) =& yx(1-x) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}(0)\cos\left(\pi t\sqrt{n^2+m^2}\right)\sin(n\pi x)\sin(m\pi x)\\ &+ \frac{1}{10\pi^2}\Big(e^{-5\pi t} - \cos(5\pi t) + \sin(5\pi t)\Big)\sin(4\pi x)\sin(3\pi y)\,,\end{aligned}$$

where the the initial condition $u(x,y,0) = f(x,y)$ gives

$$f(x,y) = yx(1-x) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}(0)\sin(n\pi x)\sin(m\pi x)\,.$$

This can be inverted, fixing the last unknowns

$$a_{nm}(0) = 4\int_0^1\int_0^1 \Big(f(x,y) - yx(1-x)\Big)\sin(n\pi x)\sin(m\pi x)dxdy\,.$$

Problem 5

We have to solve the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} + u\,.$$

Assuming that the solution can be written as an inverse Fourier transform,

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega,t) e^{-i\omega x} d\omega\,,$$

we can Fourier transform the original equation, giving a 1st order ODE

$$\frac{\partial F}{\partial t} = (-i\omega)^2 F + 2(-i\omega)F + F = (1-i\omega)^2 F\,.$$

The solution is

$$F(\omega,t) = F(\omega,0) e^{(1-i\omega)^2 t}\,,$$

where

$$F(\omega,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i\omega x} dx\,.$$

Finally, the function $u(x,t)$ is obtained by Fourier transforming back to the original coordinates

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega,0) e^{(1-i\omega)^2 t} e^{-i\omega x} d\omega\,.$$

Note that we can simplify it further (this step is not necessary in order to get 4pts) by plugging in $F(\omega,0)$ and integrating over ω,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) \int_{-\infty}^{\infty} e^{(1-i\omega)^2 t} e^{-i\omega(x-\bar{x})} d\omega d\bar{x} \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(\bar{x})\, e^{-\frac{(x-\bar{x})(x-\bar{x}+4t)}{4t}} d\bar{x}\,. \end{aligned}$$