Adiabatic invariants

What happens if we slowly change the parameters of a system that exhibits periodic motion? For example, consider the motion of a 1D system with a potential $V(q)$. For motion bounded by $V$, the coordinate $q$ will be periodic in time. Suppose $V(q)$ depends on some small parameter $\lambda$ (for example $V = \lambda q^2$), and that we start changing $\lambda$ slowly, so that over one period $T$ the change in $\lambda$ is small enough to be neglected (but over many periods the change may add up). This is known as adiabatic change.

With $\lambda = \lambda(t)$ we find that the energy of the system is not conserved

$$\dot{E} = \frac{dH}{dt} = \frac{\partial H}{\partial \lambda} \dot{\lambda} \neq 0. \quad (1)$$

Other quantities might be approximately conserved, i.e. their time-average might be constant (compared to the rate of change in $\lambda$): these are adiabatic invariants.

We will now show that action variables are adiabatic invariants. Consider the following 1D system:

$$H = \frac{p^2}{2m} + V(q, \lambda(t)). \quad (2)$$

For a fixed energy $H = E$ the system is described by periodic motion (energy can be taken to be fixed during a period since change in $\lambda$ can be neglected). The curve that the system takes can be obtained by solving for the momentum in the above equation,

$$p = p(E, \lambda, q) = \sqrt{2m(E - V)}. \quad (3)$$

And the time-average of a function $A$ over a period is defined as

$$\overline{A} = \frac{1}{T} \int_0^T A dt. \quad (4)$$

What we want to prove is that $J = \oint pdq$ is approximately constant under a period. That is, the change in $J$ adds up to zero when averaged over a period,

$$\overline{\dot{J}} = \frac{dJ}{dt} = 0. \quad (5)$$

To do this we regard the action variable as a function of energy and the slowly changing parameter $J = J(E, \lambda)$. The time dependence only enters implicitly through these two variables,

$$\dot{J} = \frac{\partial J}{\partial E} \dot{E} + \frac{\partial J}{\partial \lambda} \dot{\lambda}. \quad (6)$$
The use of partial derivatives imply that we consider \( E \) to be independent of \( \lambda \).
To be more precise, adiabatic change already implies that \( \dot{\lambda} \) and \( \dot{E} \) are small, and what we want to show is that \( \dot{J} \) is even smaller (negligible compared to the change in \( \lambda \)).

Using the relation between momentum and velocity (valid for this simple system) \( p = m \dot{q} \) we can obtain the infinitesimal time element in terms of \( dq \),

\[
\frac{dt}{dp} = \sqrt{\frac{1}{2E - V}} dq = \sqrt{2m} \frac{\partial \sqrt{E - V}}{\partial E} dq = \frac{\partial p}{\partial E} dq, \tag{7}
\]

where we twice used the parametrization of \( p \) obtained in eqn (3). Integrating this over a period gives

\[
T = \int_{0}^{T} dt = \int \frac{\partial p}{\partial E} dq \approx \frac{\partial}{\partial E} \int p dq = \frac{\partial J}{\partial E}. \tag{8}
\]

Here the second equality is only approximate, we have ignored a small change in the integration path in pulling out the derivative outside the integral. (We are justified in ignoring this small effect since the above relation will be plugged into eqn (6) where it will multiply the small quantity \( \dot{E} \), making it a second order effect.)

Next we want to relate the second term in eqn (6) to something more convenient,

\[
\frac{\partial J}{\partial \lambda} = \frac{\partial}{\partial \lambda} \int p dq \approx \int \frac{\partial p}{\partial \lambda} dq + \int_{0}^{T(\lambda)} \frac{\partial p}{\partial \lambda} \frac{\partial H}{\partial p} dt, \tag{9}
\]

where we similarly have ignored a small change in the integration path in moving the derivative inside the integration, and in the last step we used the equality \( dq = \dot{q} dt = \frac{\partial H}{\partial p} dt \). Using the fact that we assumed \( E \) and \( \lambda \) to be independent (during a fixed period) we have the following relation,

\[
0 = \frac{dE}{d\lambda} = \frac{dH}{d\lambda} = \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial \lambda}, \tag{10}
\]

where the last term is the one that we need in eqn (9). Combining the relations we obtain

\[
\frac{\partial J}{\partial \lambda} = - \int_{0}^{T(\lambda)} \frac{\partial H}{\partial \lambda} dt. \tag{11}
\]

After some effort we can now insert eqns (8) and (11) into eqn (6), and we get

\[
\dot{J} = T \dot{E} - \dot{\lambda} \int_{0}^{T(\lambda)} \frac{\partial H}{\partial \lambda} dt = \left[ T \frac{\partial H}{\partial \lambda} - \int_{0}^{T(\lambda)} \frac{\partial H}{\partial \lambda} df \right] \dot{\lambda}, \tag{12}
\]

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where in the second step we used that \( \dot{E} = \frac{\partial H}{\partial \lambda} \). We are almost done! If we can pull out the factor \( \frac{\partial H}{\partial \lambda} \) from the integral the two terms will cancel. However, this factor is not constant, so we have to do something different. Instead we integrate the whole expression over one period, and we get

\[
\int_0^T \left\{ \left[ T \frac{\partial H}{\partial \lambda} - \int_0^T \frac{\partial H}{\partial \lambda} dt \right] \dot{\lambda} \right\} dt = \left[ T \int_0^T \frac{\partial H}{\partial \lambda} dt - \int_0^T dt \int_0^T \frac{\partial H}{\partial \lambda} dt \right] \dot{\lambda} = 0, \tag{13}
\]

where we have assumed that \( \dot{\lambda} \) is constant over one period (adiabatic change). This means that \( \frac{\partial H}{\partial \lambda} \) is the only part of the integrand that depends on time, and after integration over a period the two terms cancel. Thus we have arrived at what we wanted to prove,

\[
\overline{J} = \frac{1}{T} \int_0^T \dot{J} dt = 0. \tag{14}
\]

**Example: Pendulum**

Consider a pendulum consisting of a mass \( m \) attached to a string of length \( \ell = \ell(t) \), which varies slowly in time. Also, for simplicity assume that the oscillations of the pendulum are such that the displaced angle \( \theta \ll 1 \) is small.

For fixed \( \ell \) we have the energy

\[
E = H = \frac{p_\theta^2}{2m\ell^2} + \frac{mg\ell}{2} \theta^2. \tag{15}
\]

The phase portraits are ellipses,

\[
1 = \frac{p_\theta^2}{p_{\theta 0}^2} + \frac{\theta^2}{\theta_0^2}, \tag{16}
\]

with semi-major/minor axes \( \theta_0 = \sqrt{2E/(mg\ell)} \) and \( p_{\theta 0} = \sqrt{2m\ell^2E} \). The action variable is then the area of an ellipse,

\[
J = \oint p_\theta d\theta = \pi \theta_0 p_{\theta 0} = 2\pi E \sqrt{\frac{\ell}{g}}. \tag{17}
\]

The fact that \( J \) is an adiabatic invariant thus implies that

\[
\overline{E\sqrt{\ell}} = \text{constant}. \tag{18}
\]

That is, the energy of the pendulum is on average proportional to the inverse square root of the length

\[
E \sim \ell^{-1/2}. \tag{19}
\]
From this we can deduce that the semi-major/minor axes of the phase portrait changes on average as follows:

\[
\frac{1}{\theta_0} \sim p_{\theta_0} \sim \ell^{3/4},
\]  

meaning that the eccentricity of the ellipsis changes with time, while the area stays the same.