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Lektion 1

1. The Hamiltonian of a relativistic particle om mass m has the form

$$H = \sqrt{m^2c^4 + \mathbf{p}^2c^2} \ ,$$

where $\mathbf{p}$ is the 3-dimensional momentum and c is the speed of light. Find a Lagrangian corresponding to this Hamiltonian.

2.A dynamical system has the Lagrangian

$$\begin{aligned} a) \quad L &= \frac{5}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 + \dot{q}_1\dot{q}_2 cos(q_1 - q_2) + 3cos(q_1) + cos(q_2). \\ b) \quad L &= \frac{1}{2}\left[(\dot{q}_1 - \dot{q}_2)^2 + a\dot{q}_1^2 t^2\right] - acos(q_2). \end{aligned} \qquad (1)$$

Find a Hamiltonian corresponding to this Lagrangian.

3. A dynamical system has the Hamiltonian

$$H = p_1p_2 + q_1q_2 \ .$$

Find a Lagrangian corresponding to this Hamiltonian.

4. A Hamiltonian for a system with one degree of freedom given by

$$H = \frac{p^2}{2a} - bqpe^{-\alpha t} + \frac{ab}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2} \ ,$$

where a, b, α and k are constants.
a) Find a Lagrangian corresponding to this Hamiltonian.
b) Find an equivalent Lagrangian that does not depend on time explicitly.
c) What is the Hamiltonian corresponding to the second Lagrangian and what is the relationship between the two Hamiltonians?

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Lektion 2

1. A dynamical system has the Hamiltonian

$$H = q_1p_2 - q_2p_1 + a(p_1^2 + p_2^2) ,$$

Find a Lagrangian corresponding to this Hamiltonian.

2. A dynamical system has the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2(p_2^2 + q_2^2) + q_1^2\right)$$

Derive the canonical equations and find their general solution.

3. A dynamical system has the Lagrangian

$$L = \frac{mR^2}{2}\left(\dot{\theta}^2 + \frac{\dot{\phi}^2}{sin^2\theta}\right) - mgRcos\theta .$$

Find a Hamiltonian corresponding to this Lagrangian. Find a cyclic variable and reduce the problem to a family of problems with one degree of freedom. Draw the phase portraits for the 1-dimensional systems.

4. Homework A point-particle of mass m moves in a 2-dimensional space in the central potential

$$V(r) = -q/r.$$

Find a Hamiltonian for the system using polar coordinates. Find a cyclic variable and reduce the problem to a family of one-dimensional problems. Draw the phase portraits.

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Lektion 3

1. The length l of a mathematical pendulum varies with time $l = l(t)$. Derive a Lagrangian and a Hamiltonian for the system.

2. Find a canonical transformation defined by the following generating functions:

$$a)F = ln(qt)e^P \ , \qquad b)F = qln(P) \ .$$

3.Homework A dynamical system has the Hamiltonian

$$H = \frac{1}{2}q^2 + at^2q^2 - 2tpq + \frac{1}{a}p^2 \ ,$$

where a is a constant. Apply a canonical tansformation defined by the generating function

$$F = \frac{1}{2}atq^2 - qP \ .$$

Find the new Hamiltonian in terms of P and Q.

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Lektion 4

1. A one-dimensional free point-particle is described by the Hamiltonian $H = p^2/2m$. Find a time-dependent canonical transformation such that the new momentum P coincides with H. Give a general solution of the equations of motion sing this canonical transformation.

2. A change of variables in the phase-space plane is defined by the equations

$$P = p\ cos\ t + q\ sin\ t\ , \quad Q = q\ cos\ t - p\ sin\ t\ .$$

Show that this is a canonical transformation. Find the generating function $F(q, P, t)$.

3.Homework A dynamical system has the Hamiltonian $H = pq^3/2t$. Find a new Hamiltonian after the canonical transformations

$$P = pq^3\left(1 + t\ exp(\frac{1}{q^2})\right)\ , \quad Q = -\frac{1}{2}\left(\frac{1}{q^2} + ln(\frac{pq^3}{t})\right)\ .$$

4 Find the Poisson bracket of the following functions on phase-space:

$$\Phi = p^2 + q^2\ , \quad \Psi = arctg(\frac{p}{q})\ .$$

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Lektion 5

1. A canonical transformation is given by the generating function

$$F(q, P) = P^2 + ln(P + q) \ .$$

a Find the new variables P and Q as functinons of p and q.
b Verify that the new variables have the correct Poisson brackets.

2. A dynamical system has the Hamiltonian $H = p_1p_2 + q_1q_2$. Show that the functions

$$\Phi_1 = p_1^2 + q_2^2 \ , \quad \Phi_2 = p_2^2 + q_1^2$$

are conserved quantities.

3.Homework Show that the functions

$$\Phi_1 = \frac{p_1}{q_2}, \qquad \Phi_2 = (p_2 - q_2)e^{-t} \ .$$

are conserved quantities for the Hamiltonian $H = p_1q_1 - p_2q_2 + q_2^2$. Find one more integral of the motion using Poisson brackets. Find the general solution of the equations of motion using the conserved quantities.

4 Using Poisson brackets, show that the components of the Laplace-Runge-Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r} \ ,$$

are conserved quantities in the Kepler problem with Hamilonian

$$H = \frac{\mathbf{p}^2}{2m} - \frac{k}{r} \ .$$

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Lektion 6

1 The Hamiltonian of a system has the form

$$H = \frac{1}{2}(\frac{1}{q^2} + p^2 q^4)$$

Find the equations of motion for q. Find a canonical transformation which reduces H to the form of a harmonic oscillator.

2 Find Hamilton's principal function for a one-dimensional free particle of mass m. Derive the general solution of the equations of motion $p(t)$ and $q(t)$.

3 Suppose the potential in a problem with one degree of freedom is linearily dependent on time, such that the Hamiltonian has the form

$$H = \frac{p^2}{2m} - mAtx ,$$

where A is a constant. Solve the dynamical problem by means of Hamilton's principal function, under the initial conditions $t = 0, x = 0, p = mv_0$.

4 The Hamiltonian of a system has the form

$$H = \frac{1}{2}e^t \left(\frac{p^2}{2m} + Aq \right) ,$$

Find the general solution for $p(t)$ and $q(t)$ using Hamilton's principal function.

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Lektion 7

1. A mechanical system has two angle-like degrees of freedom , $0 \leq \theta \leq \pi$, $-\pi \leq \phi \leq \pi$. The Lagrangian for the system is

$$L = \frac{I}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 cos^2\theta\right) \ .$$

Find a complete solution of the Hamilton-Jacobi equation in terms of 1-dimensional integrals.

2. A Hamiltonian system with two degrees of freedom is defined by the Hamiltonian

$$H = \frac{1}{2}(p_1 q_2 + 2 p_1 p_2 + q_1^2) \ .$$

Find a complete solution of the Hamilton-Jacobi equation. Find the general solution for $q_1(t)$ and $q_2(t)$.

3.Homework A Hamiltonian system with two degrees of freedom is defined by the Hamiltonian

$$H = e^t \frac{p_2 + q_2}{p_1 + q_1} \ .$$

a Find a complete solution of the Hamilton-Jacobi equation.
b Find the general solution for $q_1(t)$ and $q_2(t)$.

4 A two-dimensional mechanics is defined by the lagrangian

$$L = \frac{\dot{x}^2}{2y^2} + \frac{\dot{y}^2}{2} - \alpha x^2 y^2 \ .$$

Find a complete solution of the Hamilton-Jacobi equation in terms of one-dimensional integrals.

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Analytical Mechanics MN part 2 and fk

Lesson 8

1. A particle of mass m is moving in the presence of two attracting centres placed at the points $x = -a$, $y = 0$ and $x = +a$, $y = 0$ (see Fig. 1). The potentials are $-\frac{A}{r_1}$ and $-\frac{B}{r_2}$ respectively. Find the principal function of Hamilton in terms of one-dimensional integrals. Use the coordinates $\xi = \frac{r_1+r_2}{2}$ and $\eta = \frac{r_1-r_2}{2}$. (The variables will not separate if you use Cartesian coordinates or the coordinates r_1 and r_2.)

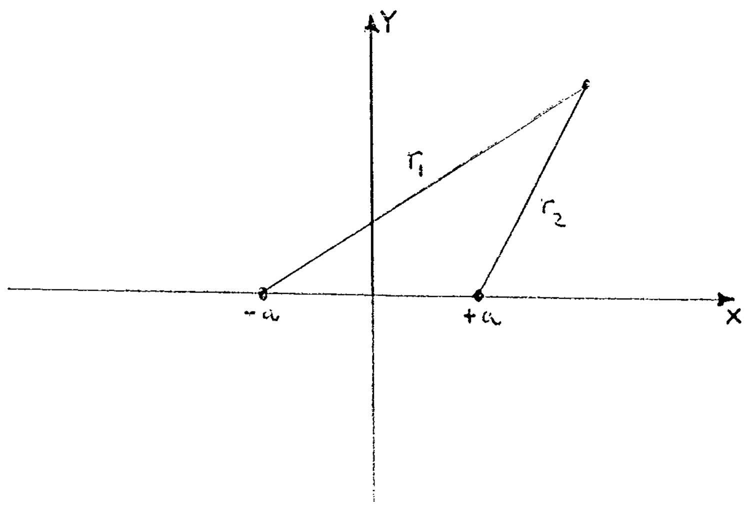
2. A Hamiltonian system with two degrees of freedom is defined by the Hamiltonian

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}\left(\frac{1}{2}p_2^2 + \frac{1}{2}q_2^2\right)^2 q_1^2. \tag{1}$$

a) Find a complete solution of the Hamilton-Jacobi equation in terms of one-dimensional integrals.

b) Draw the phase portraits on the phase planes (p_1, q_1) and (p_2, q_2).

c) Introduce action-angle variables and express the Hamiltonian as a function of action variables $J_1$and J_2. Find frequencies ν_1 and ν_2 as functions of the action variables.



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Analytical Mechanics MN part 2 and fk

Lesson 9

1. An elastic ball is oscillating between two very slowly moving walls (see Fig. 1).

a) Find an action variable in this system.

b) How the velocity of the ball depends on the distance $L(t)$ between the walls?

2. A plane pendulum of small amplitude is constrained to move on an inclined plane (see Fig. 2). How does the amplitude change when the inclination angle α of the plane is changed slowly?

3. To the lowest order in correction terms, the relativistic Hamiltonian for the one-dimensional harmonic oscillator has the form

$$H = \frac{1}{2m}(p^2 + m\omega^2 q^2) - \frac{p^4}{8m^3c^2}.$$

Calculate the lowest order relativistic correction to the frequency of the harmonic oscillator.

4. A plane isotropic harmonic oscillator is perturbed by a change in the Hamiltonian of the form,

$$\Delta H = \varepsilon p_x^2 p_y^2.$$

Find the shifts in frequencies in the first order in ε.

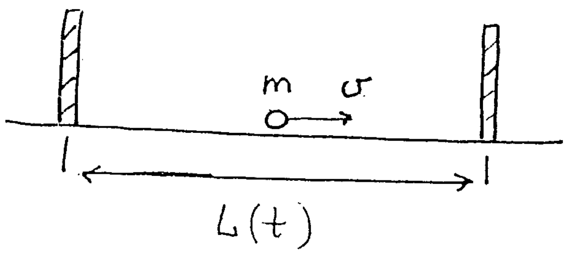

Fig. 1

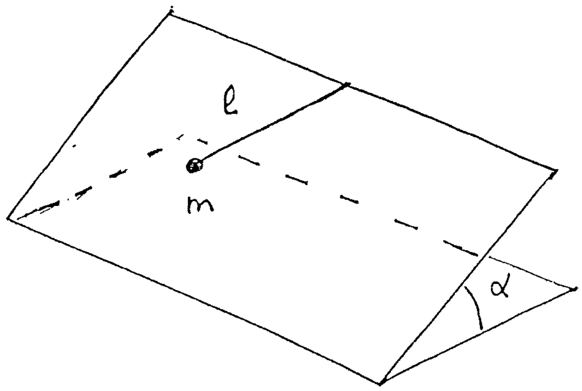

Fig. 2