Time-independent perturbation theory

Suppose we have an unperturbed 1D integrable system expressed in action-angle variables,

$$H_0(\varphi_0, J_0) = H_0(J_0) \quad \text{(by definition)}\,. \tag{1}$$

As a consequence

$$\dot{\varphi}_0 = \frac{\partial H_0}{\partial J_0} \equiv \nu_0 = \text{const.} \qquad \Rightarrow \qquad \varphi_0 = \nu_0\, t + \delta_0\,. \tag{2}$$

After a perturbation has been introduced, (φ_0, J_0) remain canonical variables,

$$H = H_0(J_0) + \epsilon H_1(\varphi_0, J_0) + \epsilon^2 H_2(\varphi_0, J_0) + \dots\,, \tag{3}$$

however, they are no longer action-angle variables.

Let us find a canonical transformation from (φ_0, J_0) to (φ, J), so that J is the action variable for the perturbed system (i.e. H) and φ is the corresponding angle variable $\varphi = \nu\, t + \delta$. We thus want to re-express the Hamiltonian as

$$H = H(J, \epsilon)\,, \tag{4}$$

which we write as

$$H(J, \epsilon) = \alpha(J, \epsilon) = \alpha_0(J) + \epsilon\, \alpha_1(J) + \epsilon^2\, \alpha_2(J) + \dots\,. \tag{5}$$

Consequently the frequency has the expansion

$$\nu = \frac{\partial H}{\partial J} = \frac{\partial \alpha_0}{\partial J} + \epsilon \frac{\partial \alpha_1}{\partial J} + \epsilon^2 \frac{\partial \alpha_2}{\partial J} + \dots\,, \tag{6}$$

where the first term is the unperturbed frequency $\frac{\partial \alpha_0}{\partial J} = \nu_0$.

We seek a generating function for the canonical transformation on the form

$$Y(\varphi_0, J, \epsilon) = \varphi_0 J + \epsilon\, Y_1(\varphi_0, J) + \epsilon^2\, Y_2(\varphi_0, J) + \dots\,, \tag{7}$$

and it has to satisfy the Hamilton-Jacobi equation

$$H\Big(\varphi_0, \frac{\partial Y}{\partial \varphi_0}, \epsilon\Big) = \alpha(J, \epsilon)\,. \tag{8}$$

The general approach is to expand the Hamilton-Jacobi equation in ϵ and derive the equations for the expansion coefficients α_i and Y_i . In this lecture we will only work to first order in ϵ (for second and higher orders see Goldstein). We have already expanded the energy $\alpha(J,\epsilon)$ in equation (5), so let us focus on the LHS

$$\begin{aligned} H\Big(\varphi_0, \frac{\partial Y}{\partial \varphi_0}, \epsilon\Big) &= H_0\Big(\frac{\partial Y}{\partial \varphi_0}\Big) + \epsilon H_1\Big(\varphi_0, \frac{\partial Y}{\partial \varphi_0}\Big) + \dots \\ &= H_0(J) + \epsilon \frac{\partial H_0}{\partial J}\frac{\partial Y_1}{\partial \varphi_0} + \epsilon H_1(\varphi_0, J) + \dots\,, \end{aligned} \tag{9}$$

where in the second step we plugged in the expansion of $Y(\varphi_0, J, \epsilon)$ to first order. Identifying LHS and RHS terms of same order the Hamilton-Jacobi equation gives

$$\begin{aligned} &\alpha_0(J) = H_0(J)\,, \\ &\alpha_1(J) = \frac{\partial H_0}{\partial J}\frac{\partial Y_1}{\partial \varphi_0} + H_1(\varphi_0, J) = \nu_0 \frac{\partial Y_1}{\partial \varphi_0} + H_1(\varphi_0, J)\,, \\ &\alpha_2(J) = \dots\,. \end{aligned} \tag{10}$$

The first equation is trivially identifying α_0 with the unperturbed Hamiltonian (with J_0 replaced by J). The next equation relates α_1 with two known functions ν_0 and H_1 and the unknown function Y_1.

We can still do better, and solve for α_1 even without knowing Y_1. Recall that $\alpha(J)$ is required to be a constant of motion, and that the original phase space coordinates (p, q) should be periodic in both φ_0 and φ with the same period. The latter property implies that the generating function coefficients Y_j are periodic functions of φ_0, and so we can Fourier series expand them,

$$Y_j(\varphi_0, J) = \sum_{k\in\mathbb{Z}} B_k^{(j)}(J)\, e^{2\pi i\, k \varphi_0}\,. \tag{11}$$

This implies that $\frac{\partial Y_j}{\partial \varphi_0}$ has no constant term (the derivative kills it), and so if we average it over one period we get zero, and thus we can eliminate $\frac{\partial Y_1}{\partial \varphi_0}$ from the above 1st order equation. The contribution from the average of $H_1(\phi_0, J)$ is easily calculated from the time-dependence of $\varphi_0 = \nu_0\, t + \delta_0$,

$$\alpha_1(J) = \overline{H_1(\varphi_0, J)} = \frac{1}{T}\int_0^T H_1(\varphi_0, J) dt\,, \tag{12}$$

which gives a known function. Plugging this relation back into the 1st order equation (without averaging) we obtain an equation for Y_1,

$$\nu_0 \frac{\partial Y_1}{\partial \varphi_0} = \overline{H_1(\varphi_0, J)} - H_1(\varphi_0, J)\,, \tag{13}$$

where again everything is known except for Y_1, and thus Y_1 can be integrated

$$Y_1 = \int \frac{\overline{H_1(\varphi_0, J)} - H_1(\varphi_0, J)}{\nu_0} d\varphi_0\,. \tag{14}$$

We can also do the integral explicitly in terms of the Fourier coefficients on both sides,

$$B_k^{(1)}(J) = -\frac{c_k(J)}{2\pi i\, k\nu_0}\,, \qquad k \neq 0 \tag{15}$$

where $c_k(J)$ are the Fourier coefficients of $H_1(\varphi_0, J) = \sum_{k\in\mathbb{Z}} c_k(J)\, e^{2\pi i\, k\varphi_0}$. Note that the constant $B_0^{(1)}(J)$ is not determined, but this is just an arbitrary integration constant that does not change the canonical transformation, thus we may as well set $B_0^{(1)}(J) = 0$.

Degenerate systems and resonances

We can immediately generalize the 1D time-independent perturbation theory to $2n$ degrees of freedom, thus we assume we have n pairs of "old" action-angle variables (φ_{0i}, J_{0i}) and correspondingly n pairs of "new" action-angle variables (φ_i, J_i). In this case we have the equations for α,

$$\begin{aligned} \alpha_0(J_i) &= H_0(J_i)\,, \\ \alpha_1(J_i) &= \overline{H_1(\varphi_{0i}, J_i)}\,, \\ \alpha_2(J_i) &= \ldots\,. \end{aligned} \tag{16}$$

and the equations for Y,

$$\begin{aligned} Y_0(\varphi_{0i}, J_i) &= \varphi_{0i} J_i\,, \\ \nu_{0i}\frac{\partial Y_1}{\partial \varphi_{0i}} &= \overline{H_1(\varphi_{0i}, J_i)} - H_1(\varphi_{0i}, J_i)\,. \end{aligned} \tag{17}$$

Unlike the 1D case we cannot directly integrate the last equation to find Y_1; however, the Fourier expansion coefficients can still be solved like before,

$$B_{\bar{k}}^{(1)}(J) = -\frac{c_{\bar{k}}(J)}{2\pi i\, \nu_0\, \bar{k}\cdot\bar{\nu}_0}\,, \qquad \bar{k} \neq 0 \tag{18}$$

where $c_{\bar{k}}(J)$ are the Fourier coefficients of $H_1 = \sum_{\bar{k}\in\mathbb{Z}} c_{\bar{k}}(J)\, e^{2\pi i\, \bar{k}\cdot\bar{\varphi}_0}$ and similarly for the generating function we have $Y_j = \sum_{\bar{k}\in\mathbb{Z}} B_{\bar{k}}^{(j)}(J)\, e^{2\pi i\, \bar{k}\cdot\bar{\varphi}_0}$.

In this case the solution for the Fourier coefficients is clearly problematic if the denominator happens to vanish

$$\bar{k}\cdot\bar{\nu}_0 = k_i \nu_{0i} = 0\,. \tag{19}$$

Indeed this happens in degenerate systems (where frequencies are commensurate). When $\bar{k}\cdot\bar{\varphi}_0 = 0$ we have a resonance, and this has to be dealt with an appropriate manner; however, this goes beyond the scope of this course (see Goldstein 12.4 for a brief discussion).

Example

Consider the 1D anharmonic oscillator as a perturbation of the harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2m}\left[p^2 + m^2\omega_0^2 q^2\left(1 + \epsilon\frac{q}{q_0}\right)\right] \equiv H_0 + \epsilon H_1\,, \tag{20}$$

where

$$\begin{aligned} \omega_0 &= 2\pi\nu_0 = 2\pi\sqrt{\frac{k}{m}}\,,\\ H_1 &= \frac{m\omega_0^2 q^3}{2q_0}\,, \end{aligned} \tag{21}$$

and q_0 is the maximum amplitude of the unperturbed oscillator (for some energy). Using the action-angle variables for the harmonic oscillator (see lecture 6) we have

$$\begin{aligned} H_0 &= J_0\nu_0\,,\\ H_1 &= \frac{m\omega_0^2}{2q_0}\left(\frac{J_0}{\pi m\omega_0}\right)^{3/2}\sin^3\varphi_0\,, \end{aligned} \tag{22}$$

where $\varphi_0 = \nu_0 t + \delta_0$.

We series expand $\alpha(J) = \alpha_0(J) + \epsilon\alpha_1(J) + \ldots$ with

$$\begin{aligned} \alpha_0(J) &= J_0\nu_0\,,\\ \alpha_1(J) &= \overline{H}_1 = \frac{1}{T}\frac{m\omega_0^2}{2q_0}\left(\frac{J_0}{\pi m\omega_0}\right)^{3/2}\int_0^T \sin^3\varphi_0 dt = 0\,, \end{aligned} \tag{23}$$

where the integral vanishes because we are averaging over an odd function, and thus the first order perturbation of α vanishes. In Goldstein 12:4 it is shown that

$$\alpha_2(J) = -\frac{15J^2}{64\pi^2 m q_0^2}\,, \tag{24}$$

which leads to the perturbed frequency

$$\nu = \frac{\partial\alpha}{\partial J} = \frac{\partial\alpha_0}{\partial J_0} + \epsilon^2\frac{\partial\alpha_2}{\partial} = \nu_0 - \frac{15J}{32\pi^2 m q_0^2}\epsilon^2 + \ldots\,. \tag{25}$$

This can be equivalently written as

$$\frac{\Delta\nu}{\nu_0} = -\frac{15}{16}\epsilon^2 + \ldots\,, \tag{26}$$

where we used that $q_0^2 = \frac{E}{2\pi^2\nu_0^2 m} = \frac{J}{2\pi^2\nu_0 m}$ to lowest order in ϵ.