The number theory of superstring amplitudes

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October 2015

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Abstract

The following article is intended as a survey of recent results at the interface of number theory and superstring theory. We review the expansion of scattering amplitudes – central observables in field and string theory – in the inverse string tension where elegant patterns of multiple zeta values occur. More specifically, the Drinfeld associator and the Hopf algebra structure of motivic multiple zeta values are shown to govern tree-level amplitudes of the open superstring. Partial results on the quantum corrections are discussed where elliptic analogues of multiple zeta values play a central rôle.

Contents

1 Introduction

Around 1970, string theory was born out of an attempt to describe pion scattering, see [1] for a recent historic account. Even though the rôle of string theory has changed a lot over the past 45 years – most notably from a model of hadrons and mesons to a candidate framework for quantum gravity – its scattering amplitudes have been of constant interest. On the one hand, they provide fertile testing grounds for string dualities [2] or possible phenomenological signatures of string theory [3–5] in connection with a low string scale [6,7]. On the other hand, string amplitudes are a prominent tool to obtain a novel viewpoint on interacting quantum field theories and perturbative gravity which arise in the limit where strings shrink to point particles. In many instances, the hidden simplicity of and relations between gauge-theory and gravity amplitudes are invisible to conventional methods (Lagrangians or Feynman diagrams) but follow naturally from string theory, see for instance [8–11].

In this work, we review recent encounters of string amplitudes with modern topics in number theory. In the "tree-level" approximation, open-string amplitudes depend on the strings' fundamental length scale through iterated integrals in the unit interval and therefore involve multiple zeta values (MZVs). As we will see, the rich mathematical patterns of the MZVs' appearance can be understood from the Drinfeld associator [12, 13] and the Hopf algebra structure of motivic MZVs [14]. We also report on tree-level amplitudes of the closed-string [14, 15] as well as recent results [16] on the leading quantum corrections, "one-loop amplitudes". In the open-string sector, the latter are governed by iterated integrals on a genus-one surface and thus involve elliptic analogues of MZVs as studied by Enriquez [17, 18].

1.1 The disk amplitude

Tree-level scattering amplitudes of open superstring states are given by iterated integrals along the boundary of a disk. The integrand is a correlation function of vertex operators which insert the degrees of freedom of the external states onto a worldsheet of disk topology. Using the pure spinor formulation of the superstring [19], the correlator has been evaluated recently for any number of massless external legs [20],

$$A(1,2,\ldots,n;\alpha') = \sum_{\sigma\in S_{n-3}} F^\sigma(s_{ij}) A_{\rm YM}(1,\sigma(2,3,\ldots,n{-}2),n{-}1,n) \ , \tag{1.1}$$

where the labels $1,2,\ldots,n$ on the left hand side refer to the polarizations and momenta of the external gauge bosons or their supersymmetry partners. Their ordering specifies a cyclic arrangement of punctures along the disk boundary, and the additional argument α' denotes the inverse string tension or the squared string length scale. On the right hand side, $A_{\rm YM}(1,\sigma(2,3,\ldots,n{-}2),n{-}1,n)$ are partial tree amplitudes in the super Yang-Mills theory obtained in the point particle limit $\alpha'\to 0$ [8]. They encode sums of Feynman diagrams obtained in degeneration limits of the disk worldsheet (see figure 1) and also depend on the external states in a cyclic ordering which is governed by $(n-3)!$ permutations $\sigma\in S_{n-3}$.

The objects of central interest to this work are the integrals $F^\sigma(s_{ij})$ in (1.1), we will report on the results of [13, 14] on their expansion in α'. In a parametrization of the disk boundary through

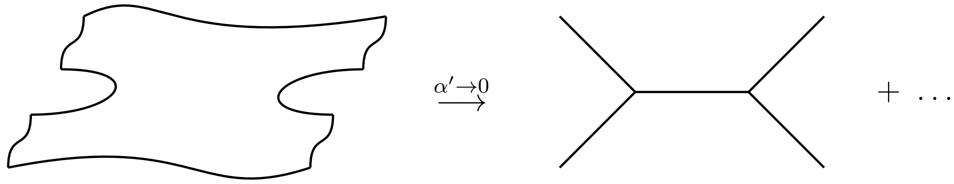

Figure 1: The disk worldsheet describing open-string scattering at tree level degenerates to Feynman diagrams in the point-particle or field-theory limit $\alpha' \to 0$, where the ellipsis refers to further representatives of Feynman diagrams.

real coordinates $z_j \in \mathbb{R}$ with $z_{ij} \equiv z_i - z_j$ [20],

$$F^\sigma(s_{ij}) \equiv (-1)^{n-3} \int\limits_{0\leq z_2 \leq z_3 \leq \ldots \leq z_{n-2} \leq 1} \mathrm{d}z_2\, \mathrm{d}z_3\, \ldots\, \mathrm{d}z_{n-2} \left(\prod_{i<j}^{n-1} |z_{ij}|^{s_{ij}} \right) \sigma \Big\{ \prod_{k=2}^{n-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \Big\} . \tag{1.2}$$

We have fixed the SL(2) symmetry on the disk by choosing $z_1 = 0, z_{n-1} = 1$ and $z_n = \infty$. The permutation $\sigma \in S_{n-3}$ is understood to act on the labels $2, 3, \ldots, n-2$ in the curly bracket while leaving $\sigma(1) = 1$. The integrals in (1.2) carry the entire α'-dependence of the disk amplitude through dimensionless combinations

$$s_{ij} \equiv \alpha'(k_i + k_j)^2 \tag{1.3}$$

of the external momenta k_i which are vectors of the D-dimensional Lorentz group. Momentum conservation $\sum_{i=1}^n k_i = 0$ and the on-shell condition $(k_i)^2 = 0$ for massless particles leave $\frac{n}{2}(n-3)$ independent Mandelstam variables s_{ij}. As we will demonstrate, the integrals in (1.2) reduce as follows in the field-theory limit $\alpha' \to 0$,

$$\lim_{\alpha' \to 0} F^\sigma(s_{ij}) = \begin{cases} 1 \;:\; \sigma(2,3,\ldots,n-2) = 2,3,\ldots,n-2 \\ 0 \;:\; \text{otherwise} \end{cases} , \tag{1.4}$$

i.e. their Taylor expansion w.r.t. s_{ij} in (1.3) encodes the string-corrections to super Yang-Mills theory. The expansion w.r.t. s_{ij} and thereby α' turns out to exhibit uniform transcendentality[1]: The w'th order in α' is accompanied by MZVs of transcendental weight w.

In the following sections, we will describe two organizing principles underlying the α'-expansion

[1]The terminology here and in later places relies on the commonly trusted conjectures on the transcendentality of MZVs.

of the $F^\sigma(s_{ij})$. More specifically,

- A matrix representation of the Drinfeld associator generates the Taylor expansion in s_{ij} in a recursive manner w.r.t. the multiplicity n [13], see section 2.
- Motivic MZVs and their Hopf algebra structure allow to extract the complete information on $F^\sigma(s_{ij})$ from its coefficients along with primitive MZVs ζ_w [14], see section 3.

In section 4, we conclude with a brief discussion of generalizations to closed strings or quantum corrections and raise open questions.

2 The α'-expansion from the Drinfeld associator

In this section, we review the recursion in [13] to obtain α'-expansion of the integrals in (1.2) from the Drinfeld associator [21, 22]. This is achieved by establishing a Knizhnik-Zamolodchikov (KZ) equation for a deformation of the integrals in question through an auxiliary worldsheet puncture z_0. Certain boundary values of the deformed integrals as $z_0 \to 0$ and $z_0 \to 1$ are found to yield the original $F^\sigma(s_{ij})$ at multiplicity $n-1$ and n, respectively. Recalling that the superscript σ denotes permutations of the legs $2, 3, \ldots, n-2$, one can write the resulting recursion as [13]

$$F^{\sigma_i} = \sum_{j=1}^{(n-3)!} \big[\Phi(e_0, e_1)\big]_{ij} \, F^{\sigma_j}\big|_{k_{n-1}=0} \ , \tag{2.1}$$

where the kinematic regime $k_{n-1} = 0$ on the right hand side gives rise to $(n-1)$-point integrals,

$$F^{\sigma(23\ldots n-2)}\big|_{k_{n-1}=0} = \begin{cases} F^{\sigma(23\ldots n-3)} & \text{if } \sigma(n{-}2) = n{-}2 \\ 0 & \text{otherwise} \end{cases} . \tag{2.2}$$

The expressions for and derivation of the matrices e_0 and e_1 will be discussed in the subsequent.

2.1 Background on MZVs and the Drinfeld associator

Before setting up the construction of the integrals $F^\sigma(s_{ij})$, we shall review the convention for MZVs and selected properties of the Drinfeld associator. MZVs of transcendental weight $w \in \mathbb{N}_0$ can be defined through iterated integrals labelled by a word in the two-letter alphabet $v_j \in \{0, 1\}$,

$$\zeta_{\{v_1 v_2 \ldots v_w\}} \equiv (-1)^{\sum_{j=1}^w v_j} \int\limits_{0 \le z_1 \le z_2 \le \ldots \le z_w \le 1} \frac{\mathrm{d}z_1}{z_1 - v_1} \, \frac{\mathrm{d}z_2}{z_2 - v_2} \, \cdots \, \frac{\mathrm{d}z_w}{z_w - v_w} \ , \tag{2.3}$$

where $v_1=1$ and $v_w=0$ ensure convergence. Divergent integrals arising for $v_1=0$ or $v_w=1$ can be addressed using the shuffle regularization prescription [23],

$$\zeta_{\{0\}}=\zeta_{\{1\}}=0\ , \qquad \zeta_{\{v\}}\cdot\zeta_{\{u\}}=\zeta_{\{v\sqcup\!\sqcup u\}}\ , \tag{2.4}$$

with the standard shuffle product $\sqcup\!\sqcup$ on words $v=v_1v_2\ldots$ and $u\equiv u_1u_2\ldots$. The representation of MZVs as nested sums can be recovered from the above integrals via

$$\zeta_{n_1,n_2,\ldots,n_r}\equiv\sum_{0<k_1<k_2<\ldots<k_r}^{\infty}k_1^{-n_1}k_2^{-n_2}\ldots k_r^{-n_r}=\zeta_{\{\underbrace{10\ldots0}_{n_1}\underbrace{10\ldots0}_{n_2}\,\cdots\cdots\,\underbrace{10\ldots0}_{n_r}\}}\ , \tag{2.5}$$

such that for example $\zeta_{\{10\}}=-\zeta_{\{01\}}=\zeta_2$.

The Drinfeld associator governs the universal monodromy of the KZ equation[2] with $z_0\in\mathbb{C}\backslash\{0,1\}$ and Lie-algebra generators e_0,e_1:

$$\frac{\mathrm{d}\hat{\mathbf{F}}(z_0)}{\mathrm{d}z_0}=\left(\frac{e_0}{z_0}+\frac{e_1}{1-z_0}\right)\hat{\mathbf{F}}(z_0)\ . \tag{2.6}$$

The solution $\hat{\mathbf{F}}(z_0)$ of the KZ equation lives in the vector space the representation of e_0 and e_1 acts upon. This general setup will later on be specialized to $(n-2)!$-component realizations of $\hat{\mathbf{F}}(z_0)$ closely related to the disk integrals F^σ.

Given the singularities of the differential operator in (2.6) as $z_0\to0$ and $z_0\to1$, non-analytic behaviour as $z_0^{e_0}$ and $(1-z_0)^{-e_1}$ has to be compensated when considering boundary values,

$$C_0\equiv\lim_{z_0\to0}z_0^{-e_0}\hat{\mathbf{F}}(z_0)\ ,\quad C_1\equiv\lim_{z_0\to1}(1-z_0)^{e_1}\hat{\mathbf{F}}(z_0)\ . \tag{2.7}$$

As a defining property of the Drinfeld associator, it relates the regularized boundary values in (2.7) via [21, 22]

$$C_1=\Phi(e_0,e_1)\,C_0\ . \tag{2.8}$$

At the same time, the Drinfeld associator in (2.8) can be written as a generating series of MZVs.

[2] The sign convention for e_1 varies in the literature.

In terms of their integral representation (2.3), we have [24]

$$\begin{aligned}\Phi(e_0,e_1) &= \sum_{v\in\{0,1\}^\times} e_{v_1}e_{v_2}\ldots e_{v_j}\ldots \zeta_{\ldots v_j \ldots v_2 v_1} \\ &= 1+\zeta_2[e_0,e_1]+\zeta_3[e_0-e_1,[e_0,e_1]]+\ldots\ .\end{aligned} \tag{2.9}$$

Hence, the Drinfeld associator plays a two-fold rôle as a generating series for MZVs in (2.9) and the universal monodromy of the KZ equation as in (2.8). Like this, it will be shown to hold the key to the recursion in (2.1) for disk integrals.

2.2 Deforming the disk integrals

In order to relate the disk integrals (1.2) to the Drinfeld associator, we will follow the lines of [25] and study a deformation that satisfies the KZ equation (2.6). In addition to an additional disk puncture $z_0 \in [0,1]$, auxiliary Mandelstam invariants $s_{02},\ldots,s_{0,n-2}\in\mathbb{R}$ as well as an integer parameter $\nu=1,2,\ldots,n-2$ are introduced in

$$\begin{aligned}\hat{F}^\sigma_\nu(s_{ij},s_{0k},z_0) \equiv (-1)^{n-3} &\int\limits_{0\leq z_2\leq z_3\leq\ldots\leq z_{n-2}\leq z_0} \mathrm{d}z_2\,\mathrm{d}z_3\,\ldots\,\mathrm{d}z_{n-2}\left(\prod_{i<j}^{n-1}|z_{ij}|^{s_{ij}}\right) \\ &\times\left(\prod_{k=2}^{n-2}|z_{0k}|^{s_{0k}}\right)\sigma\Big\{\prod_{l=2}^{\nu}\sum_{m=1}^{l-1}\frac{s_{ml}}{z_{ml}}\prod_{p=\nu+1}^{n-2}\sum_{q=p+1}^{n-1}\frac{s_{pq}}{z_{pq}}\Big\}\,.\end{aligned} \tag{2.10}$$

The integration domain for $z_2,\ldots,z_{n-2}$ reduces to the original one in (1.2) if $z_0\to 1$ and sends all integration variables to zero if $z_0\to 0$. As a consequence of the extra Mandelstam invariants s_{0k}, different values of $\nu=1,2,\ldots,n-2$ yield inequivalent integrals[3] such that the $(n-3)!$ permutations $\sigma\in S_{n-3}$ together with the range of ν yield a total of $(n-2)!$ functions in (2.10). It will be convenient to combine these objects to an $(n-2)!$-component vector whose entries are ordered as $\hat{\mathbf{F}}=(\hat{F}_{n-2},\hat{F}_{n-3},\ldots,\hat{F}_1)$.

The $(n-2)!$ components in (2.10) exceeding the number of $(n-3)!$ desired integrals in (1.2)

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[3]In the original disk integrals (1.2), rearranging the curly bracket of the integrand as

$$\prod_{l=2}^{n-2}\sum_{m=1}^{l-1}\frac{s_{ml}}{z_{ml}}\to\prod_{l=2}^{\nu}\sum_{m=1}^{l-1}\frac{s_{ml}}{z_{ml}}\prod_{p=\nu+1}^{n-2}\sum_{q=p+1}^{n-1}\frac{s_{pq}}{z_{pq}}$$

amounts to adding total derivatives w.r.t. $z_2,\ldots,z_{n-2}$ which vanish in presence of the Koba-Nielsen factor $\prod_{i<j}^{n-1}|z_{ij}|^{s_{ij}}$. Tentative boundary contributions at $z_j=z_{j\pm1}$ are manifestly suppressed by $|z_j-z_{j\pm1}|^{s_{j,j\pm1}}$ for positive real part of $s_{j,j\pm1}$ which propagates to generic complex values by analytic continuation.

are required to ensure that the deformed vector $\hat{\mathbf{F}}$ satisfies the KZ equation (2.6). Clearly, the variables e_0, e_1 therein become $(n-2)! \times (n-2)!$ matrices, and it will be illustrated by the later examples that their entries are linear in the Mandelstam variables s_{ij} as well as their auxiliary counterparts s_{0k}. Hence, the regularized boundary values (2.7) of $\hat{\mathbf{F}}$ will be related as in (2.8) by a $(n-2)! \times (n-2)!$ matrix representation of the Drinfeld associator. As is explained in more detail in [13], the components in (2.10) give rise to regularized boundary values

$$C_0\big|_{s_{0k}=0} = (F^\sigma\big|_{k_{n-1}=0}, \mathbf{0}_{(n-3)(n-3)!})^t \ , \qquad C_1\big|_{s_{0k}=0} = (F^\sigma, \ldots)^t \tag{2.11}$$

upon setting the auxiliary Mandelstam invariants s_{0k} to zero. The $(n-3)(n-3)!$ components of C_1 in the ellipsis do not need to be evaluated. In (2.11) and many subsequent equations, the dependence on s_{ij} is suppressed. With the regularized boundary values in (2.11), the relation (2.8) becomes

$$\begin{pmatrix} F^\sigma \\ \vdots \end{pmatrix} = \big[\Phi(e_0, e_1)\big]_{(n-2)!\times(n-2)!} \begin{pmatrix} F^\sigma\big|_{k_{n-1}=0} \\ \mathbf{0}_{(n-3)(n-3)!} \end{pmatrix} \tag{2.12}$$

upon taking $s_{0k} \to 0$, and the zeros in the vector on the right hand reduces the recursion (2.12) to the form given in (2.1). From the linearity of e_0 and e_1 in s_{ij} (and therefore α'), two central properties of $F^\sigma(s_{ij})$ stated above can be easily verified:

- The $\alpha' \to 0$ limit of the disk integrals in (1.4) follows from the fact that the only contribution of the associator to this order is $\Phi(e_0, e_1) = 1 + \mathcal{O}(\alpha')$.

- Uniform transcendentality follows from the expansion (2.9) of the associator where MZVs of weight w are accompanied by w powers of e_0, e_1 and, by their linearity in s_{ij}, w powers of α'.

2.3 Four- and five-point examples

In this subsection, we firstly illustrate the recursion (2.1) by examples with $n = 4, 5$ external states and secondly explain the mechanisms leading to a KZ equation for the functions in (2.10) as well as the explicit form of e_0, e_1 at various multiplicities. As a convenient shorthand, we introduce

$$X_{ij} \equiv \frac{s_{ij}}{z_{ij}} \ . \tag{2.13}$$

$\underline{n=4}$ **points:** Here, the auxiliary vector made of (2.10) has two components

$$\begin{pmatrix} \hat{F}_2^{(2)} \\ \hat{F}_1^{(2)} \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_2 \, |z_{12}|^{s_{12}} |z_{23}|^{s_{23}} z_{02}^{s_{02}} \begin{pmatrix} X_{21} \\ X_{32} \end{pmatrix} , \tag{2.14}$$

where the derivative w.r.t. z_0 introduces a factor of $\frac{s_{02}}{z_{02}}$ into the integrand[4]. Given the $SL(2)$-fixing $(z_1, z_3, z_4) = (0, 1, \infty)$, the extra dependence on z_0 can be rearranged into factors of $\frac{1}{z_{01}} = \frac{1}{z_0}$ and $\frac{1}{z_{03}} = \frac{1}{z_0 - 1}$ via partial fraction $(z_{12}z_{02})^{-1} = (z_{12}z_{01})^{-1} - (z_{01}z_{02})^{-1}$ and integration by parts:

$$0 = -\int_0^{z_0} \mathrm{d}z_2 \, \frac{\mathrm{d}}{\mathrm{d}z_2} \, |z_{12}|^{s_{12}} |z_{23}|^{s_{23}} z_{02}^{s_{02}} = \int_0^{z_0} \mathrm{d}z_2 \, |z_{12}|^{s_{12}} |z_{23}|^{s_{23}} z_{02}^{s_{02}} \left(\frac{s_{02}}{z_{02}} + \frac{s_{12}}{z_{12}} - \frac{s_{23}}{z_{23}} \right) . \tag{2.15}$$

These manipulations lead to

$$\frac{\mathrm{d}}{\mathrm{d}z_0} \hat{F}_2^{(2)} = \frac{1}{z_0} \big[(s_{12} + s_{02}) \hat{F}_2^{(2)} - s_{12} \hat{F}_1^{(2)} \big] \tag{2.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}z_0} \hat{F}_1^{(2)} = \frac{1}{1 - z_0} \big[s_{23} \hat{F}_2^{(2)} - (s_{23} + s_{02}) \hat{F}_1^{(2)} \big] , \tag{2.17}$$

which allow to read off the following 2×2 matrix representations for e_0, e_1 upon setting $s_{02} \to 0$:

$$e_0 = \begin{pmatrix} s_{12} & -s_{12} \\ 0 & 0 \end{pmatrix} , \qquad e_1 = \begin{pmatrix} 0 & 0 \\ s_{23} & -s_{23} \end{pmatrix} . \tag{2.18}$$

Given the regularized boundary values (2.11), the main result (2.1) specializes to

$$\begin{pmatrix} F^{(2)} \\ \vdots \end{pmatrix} = \big[\Phi(e_0, e_1) \big]_{2\times 2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{2.19}$$

Note that the explicit form of the matrices (2.18) renders any nested commutator $\mathrm{ad}_0^k \mathrm{ad}_1^l [e_0, e_1]$ with $k, l \in \mathbb{N}_0$ and $\mathrm{ad}_i x \equiv [e_i, x]$ proportional to the nilpotent matrix $\left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. As a consequence, the MZVs in $\big[\Phi(e_0, e_1)\big]_{2\times 2}$ can be expressed in terms of primitives ζ_w and are consistent with

$$F^{(2)} = \frac{\Gamma(1 + s_{12}) \Gamma(1 + s_{23})}{\Gamma(1 + s_{12} + s_{23})} = \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n} (-1)^n \big[s_{12}^n + s_{23}^n - (s_{12} + s_{23})^n \big] \right) , \tag{2.20}$$

[4]The derivative w.r.t. z_0 directly acts at the level of the integrand since the boundary contribution from the z_0-dependence in the upper limit is suppressed as $\lim_{z_{n-2} \to z_0} (z_0 - z_{n-2})^{s_{0,n-2}} = 0$. As before, the limit is obvious if $s_{0,n-2}$ has a positive real part and otherwise follows from analytic continuation.

see [12] for a connection with a quotient of the associator. While the expression in (2.20) is more suitable to manifest the MZV-content of the four-point amplitude as compared to (2.19), the construction of the F^σ from the associator becomes significantly more rewarding at $n \geq 5$.

<u>$n = 5$ **points:**</u> At five-points, the auxiliary vector built from (2.10) has six components,

$$\begin{pmatrix} \hat{F}_3^{(23)} \\ \hat{F}_3^{(32)} \\ \hat{F}_2^{(23)} \\ \hat{F}_2^{(32)} \\ \hat{F}_1^{(23)} \\ \hat{F}_1^{(32)} \end{pmatrix} = \int_0^{z_0} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \prod_{i<j}^{4} |z_{ij}|^{s_{ij}} \, z_{02}^{s_{02}} z_{03}^{s_{03}} \begin{pmatrix} X_{12}(X_{13}+X_{23}) \\ X_{13}(X_{12}+X_{32}) \\ X_{12}X_{34} \\ X_{13}X_{24} \\ (X_{23}+X_{24})X_{34} \\ (X_{32}+X_{34})X_{24} \end{pmatrix} . \tag{2.21}$$

Following the methods from the $n = 4$ case, the z_0-derivatives can be cast into the form (2.6) using a sequence of partial fraction relations and integrations by parts. After setting $s_{0k} \to 0$, we can read off the resulting 6×6 matrix representation (with the shorthand $s_{ijk} \equiv s_{ij} + s_{ik} + s_{jk}$):

$$e_0 = \begin{pmatrix} s_{123} & 0 & -s_{13}-s_{23} & -s_{12} & -s_{12} & s_{12} \\ 0 & s_{123} & -s_{13} & -s_{12}-s_{23} & s_{13} & -s_{13} \\ 0 & 0 & s_{12} & 0 & -s_{12} & 0 \\ 0 & 0 & 0 & s_{13} & 0 & -s_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.22}$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_{34} & 0 & -s_{34} & 0 & 0 & 0 \\ 0 & s_{24} & 0 & -s_{24} & 0 & 0 \\ s_{34} & -s_{34} & s_{23}+s_{24} & s_{34} & -s_{234} & 0 \\ -s_{24} & s_{24} & s_{24} & s_{23}+s_{34} & 0 & -s_{234} \end{pmatrix} . \tag{2.23}$$

The regularized boundary values in (2.11) then imply the following associator construction for the functions F^σ in the five-point amplitude:

$$\begin{pmatrix} F^{(23)} \\ F^{(32)} \\ \vdots \end{pmatrix} = \big[\Phi(e_0,e_1)\big]_{6\times 6} \begin{pmatrix} F^{(2)} \\ 0 \\ \mathbf{0}_4 \end{pmatrix} \tag{2.24}$$

Note that the five-point α'-expansion in (2.24) can also be obtained from the representation of $F^{(23)}$ and $F^{(32)}$ in terms of the hypergeometric functions ${}_3F_2$ [26–30].

2.4 Higher multiplicity

The techniques to establish the KZ equation of $\hat{\mathbf{F}}(z_0)$ and to determine the matrices e_0, e_1 are universal to any value of n. Expressions for e_0, e_1 are straightforward to compute and additionally take a suggestive form; the resulting instances up to $n = 9$ can be downloaded from the website [31]. While the results for $n = 6, 7$ reproduce the α'-expansions in [27,28,32] as well as [33] to the orders tested, the associator-based method firstly makes high multiplicities $n > 7$ accessible. Even though the setup in [33] based on polylogarithms does not impose any limitations on n, its growing manual effort (e.g. in the treatment of poles) suggests to preferably rely on the Drinfeld associator at large multiplicities.

3 Motivic MZVs and the α'-expansion

The main result (2.1) of the previous section together with the expressions for e_0 and e_1 in (2.18), (2.22), (2.23) as well as [31] make the s_{ij}-dependence of the disk integrals fully explicit. The MZVs originate from the Drinfeld associator as in (2.9) and carry redundancies in view of the relations over $\mathbb{Q}$ among the iterated integrals $\zeta_{\{v\}}$ with $v \in \{0,1\}^\times$. In this section, we investigate the structure of the α'-expansion once the MZVs are reduced to their conjectural bases over $\mathbb{Q}$. In a conjectural model for MZVs using non-commutative generators $f_3, f_5, f_7, \ldots$ and a commutative variable f_2 [34], the end result for F^σ is captured by the neat expression [14]

$$\left(\sum_{k=0}^{\infty} f_2^k P_{2k}\right) \sum_{n=0}^{\infty} \Big(f_3 M_3 + f_5 M_5 + f_7 M_7 + \ldots\Big)^n \,, \tag{3.1}$$

where M_w and P_w are $(n-3)! \times (n-3)!$ matrices to be specified below. Most importantly, the coefficients P_{2k} and M_{2i+1} of the primitives f_2^k and f_{2i+1} completely determine the α'-dependence along with compositions such as $f_2 f_{2i+1}$ and $f_{2i+1} f_{2j+1}$.

3.1 Matrix-valued approach to disk amplitudes

In order to see the aforementioned relations between the coefficients of various basis MZVs over $\mathbb{Q}$, it is convenient to promote the disk integrals in (1.2) to a $(n-3)! \times (n-3)!$ matrix

$$F_\tau{}^\sigma(s_{ij}) \equiv (-1)^{n-3} \int\limits_{0 \leq z_{\tau(2)} \leq z_{\tau(3)} \leq \ldots \leq z_{\tau(n-2)} \leq 1} \mathrm{d}z_2 \, \mathrm{d}z_3 \, \ldots \, \mathrm{d}z_{n-2} \left(\prod_{i<j}^{n-1} |z_{ij}|^{s_{ij}} \right) \sigma \Big\{ \prod_{k=2}^{n-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \Big\} . \tag{3.2}$$

The additional index τ refers to permutations in S_{n-3} of the integration variables $2, 3, \ldots, n-2$ and distinguishes different integration domains $0 \leq z_{\tau(2)} \leq z_{\tau(3)} \leq \ldots \leq z_{\tau(n-2)} \leq 1$. The matrix of disk integrals in (3.2) allows to simultaneously address an $(n-3)!$ family of different tree-level subamplitudes,

$$A(1, \tau(2, 3, \ldots, n-2), n-1, n; \alpha') = \sum_{\sigma \in S_{n-3}} F_\tau{}^\sigma(s_{ij}) A_{\mathrm{YM}}(1, \sigma(2, 3, \ldots, n-2), n-1, n) \ . \tag{3.3}$$

The furnish a basis of arbitrary string subamplitudes $A(\pi(1, 2, \ldots, n); \alpha')$ with $\pi \in S_n$ [10, 11] in the same way as the $A_{\mathrm{YM}}(\ldots)$ on the right hand side are a basis of field-theory subamplitudes [35].

In principle, it suffices to know a single line of (3.2) with fixed τ since the remaining entries of the matrix can be generated by relabeling of the s_{ij} and corresponding changes in σ and τ. The description of disk integrals through a square matrix $F(s_{ij})$ as in (3.2) is useful in view of matrix multiplication. Let P_w and M_w denote $(n-3)! \times (n-3)!$ matrices whose entries are degree w polynomials in Mandelstam invariants with rational coefficients, then a reduction of MZVs to their conjectural $\mathbb{Q}$-bases at weight $w \leq 8$ yields

$$\begin{aligned} F(s_{ij}) = 1_{(n-3)! \times (n-3)!} &+ \zeta_2 P_2 + \zeta_3 M_3 + \zeta_2^2 P_4 + \zeta_2 \zeta_3 P_2 M_3 + \zeta_5 M_5 \\ &+ \zeta_2^3 P_6 + \frac{1}{2} \zeta_3^2 M_3 M_3 + \zeta_7 M_7 + \zeta_2 \zeta_5 P_2 M_5 + \zeta_2^2 \zeta_3 P_4 M_3 \\ &+ \zeta_2^4 P_8 + \zeta_3 \zeta_5 M_5 M_3 + \frac{1}{2} \zeta_2 \zeta_3^2 P_2 M_3 M_3 + \frac{1}{5} \zeta_{3,5} [M_5, M_3] + \mathcal{O}(\alpha'^9) \ . \end{aligned} \tag{3.4}$$

Remarkably, the matrix product $P_2 M_3$ along with the weight-five product $\zeta_2 \zeta_3$ is determined by the coefficients P_2 and M_3 of ζ_2 and ζ_3, respectively. The different parental letters P_w, M_w for matrices of even and odd order w in α' goes back to the different nature of the associated primitives: At even weight, $\zeta_{2n} \in \mathbb{Q}\pi^{2n}$ can be reduced to powers of $\zeta_2 = \frac{\pi^2}{6}$ with rational prefactors while no relations among ζ_{2n+1} of different odd weight[5] and powers of π are known or expected. Also, only a single left-multiplicative matrix factor of P_w is seen in each term of the expansion in (3.4) and its generalization to higher weight.

The depth-two MZVs $\zeta_{3,5}$ in the last line of in (3.4) is accompanied by a matrix commutator $[M_5, M_3] = M_5 M_3 - M_3 M_5$, but its rational prefactor $\frac{1}{5}$ is less intuitive than the lower-weight counterparts. The even more dramatic proliferation of rational prefactors at weight eleven,

$$\begin{aligned} F(s_{ij})\big|_{(\alpha')^{11}} &= \zeta_{11} M_{11} + \zeta_2^4 \zeta_3 P_8 M_3 + \frac{1}{2}\zeta_3^2 \zeta_5 M_5 M_3^2 + \frac{1}{6}\zeta_2 \zeta_3^3 P_2 M_3^3 + \zeta_2 \zeta_9 P_2 M_9 + \zeta_2^2 \zeta_7 P_4 M_7 \\ &+ \zeta_2^3 \zeta_5 P_6 M_5 + \frac{1}{5}\zeta_{3,5}\zeta_3 [M_5, M_3] M_3 + \left(9\zeta_2\zeta_9 + \frac{6}{25}\zeta_2^2\zeta_7 - \frac{4}{35}\zeta_2^3\zeta_5 + \frac{1}{5}\zeta_{3,3,5}\right)[M_3, [M_5, M_3]] \ , \end{aligned} \tag{3.5}$$

calls for a systematic understanding of how the matrix commutators enter at generic weight, see [14] for the analogous expressions at weight $w \leq 16$. The required mathematical framework will be introduced in the following subsection.

3.2 Motivic MZVs

The basis MZVs over $\mathbb{Q}$ in the α'-expansion (3.4) and (3.5) have been chosen as in [38], following the guiding principle of preferring short and simple representatives. An alternative handle on the choice of basis can be obtained by switching to a conjecturally equivalent language for MZVs: a Hopf algebra comodule, which is composed from words

$$f_{2i_1+1} \ldots f_{2i_r+1} \ f_2^k, \quad \text{with} \quad r,k \geq 0 \quad \text{and} \quad i_1,\ldots,i_r \geq 1 \tag{3.6}$$

and graded by their weight $w = 2(i_1 + \ldots + i_r) + r + 2k$. The non-commutative generators f_{2i+1} of odd weight by themselves furnish a Hopf algebra, and the additional commutative variable f_2 extend it to a Hopf algebra comodule [34]. At each weight, the enumeration of all non-commutative words of the form in (3.6) yields the same basis dimension over $\mathbb{Q}$ as conjectured for MZVs of the same weight [39].

The mapping of MZVs to non-commutative words in (3.6) is slightly involved and relies on (commonly trusted) conjectures such as the exclusion of additional algebraic relations between

[5] Also, none of the odd ζ-values has been proven to be transcendental so far: the only known facts are the irrationality of ζ_3 as well as the existence of an infinite number of odd irrational ζ's [36, 37].

MZVs beyond the known double-shuffle identities. In order to circumvent the currently intractable challenges to prove the outstanding conjectures, one lifts MZVs ζ to so-called motivic MZVs $\zeta^{\mathfrak{m}}$ whose more elaborate definition [34, 40–42] will not be reviewed in the subsequent. As a key property of motivic MZVs, they obey the same shuffle and stuffle product formulæ as the MZVs, e.g. (2.4) carries over to $\zeta^{\mathfrak{m}}_{\{v\}}\zeta^{\mathfrak{m}}_{\{u\}} = \zeta^{\mathfrak{m}}_{\{v \sqcup\!\sqcup u\}}$. The transition from MZVs to their motivic counterparts, $\zeta_{n_1,\ldots,n_r} \to \zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}$, has the advantage that many of the desirable, but currently unproven facts about MZVs are in fact proven for motivic MZVs. In particular, the commutative algebra of motivic MZVs is graded by definition, and the motivic coaction, first written down by Goncharov [40] and further studied by Brown [34, 41, 43], is well-defined.

In the framework of motivic MZVs, one can construct an isomorphism ϕ of graded algebra comodules which map any $\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}$ to non-commutative words of the form (3.6), see [43] for a thorough description. Once the normalization is fixed as

$$\phi(\zeta^{\mathfrak{m}}_w) = f_w \ , \qquad f_{2k} \equiv \frac{\zeta_{2k}}{(\zeta_2)^k}\, f_2^k \ , \tag{3.7}$$

the map ϕ can be largely determined by demanding compatibility with the algebraic structures:

$$\phi(\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}\zeta^{\mathfrak{m}}_{m_1,\ldots,m_r}) = \phi(\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}) \sqcup\!\sqcup\, \phi(\zeta^{\mathfrak{m}}_{m_1,\ldots,m_r}) \tag{3.8}$$

$$\Delta\phi(\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}) = \phi(\Delta\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}) \ . \tag{3.9}$$

While the motivic coaction on the right hand side of (3.9) [40] can become combinatorically involved at higher weights, the coaction on the non-commutative words from (3.6) is given by simple deconcatenation

$$\Delta(f_2^k f_{i_1} f_{i_2} \ldots f_{i_r}) = \sum_{j=0}^{r} (f_2^k f_{i_1} f_{i_2} \ldots f_{i_j}) \otimes (f_{i_{j+1}} \ldots f_{i_r}) \ , \qquad i_j \in 2\mathbb{N}+1 \ . \tag{3.10}$$

In combination with (3.9), this largely determines the ϕ-image of higher-depth MZVs such as

$$\phi(\zeta^{\mathfrak{m}}_{3,5}) = -5 f_3 f_5 \ , \qquad \phi(\zeta^{\mathfrak{m}}_{3,7}) = -14 f_3 f_7 - 6 f_5 f_5 \tag{3.11}$$

$$\phi(\zeta^{\mathfrak{m}}_{3,3,5}) = -5\, f_3 f_3 f_5 + \frac{4}{7}\, f_5 f_2^3 - \frac{6}{5}\, f_7 f_2^2 - 45\, f_9 f_2 \, , \tag{3.12}$$

see [43] for an efficient algorithm based on an infinitesimal version of the coaction.

However, the insensitivity of the coaction constraint (3.9) to primitives introduces an ambiguity of adding rational multiples of f_2^4, f_2^5 and f_{11} to the right hand sides of (3.11) and (3.12). The absence of such primitives in the above expressions reflects a specific choice of the isomorphism ϕ. It is convenient to tailor the ϕ-map to the choice of $\mathbb{Q}$-basis for motivic MZVs at weight w by suppressing f_w in the ϕ-images of all basis elements except for (3.7). The ϕ-images at weights $w \leq 16$ displayed in [14] rely on reference bases of motivic MZVs over $\mathbb{Q}$ as in [38].

3.3 Cleaning up the α'-expansion

The language of non-commutative words as in (3.6) turns out to reveal the pattern of MZVs in the α'-expansions in (3.4) and (3.5). Upon passing to a motivic version of the matrix $F(s_{ij})$ in (3.2),

$$F^{\mathfrak{m}}(s_{ij}) \equiv F(s_{ij})\big|_{\zeta_{n_1,\ldots,n_r} \rightarrow \zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}} \ , \tag{3.13}$$

the above expansions (with weights $w = 9, 10$ restored) translate into the following ϕ-image under (3.11) and (3.12):

$$\begin{aligned} \phi(F^{\mathfrak{m}}(s_{ij})) = \ & (1_{(n-3)!\times(n-3)!} + f_2 P_2 + f_2^2 P_4 + f_2^3 P_6 + f_2^4 P_8 + f_2^5 P_{10}) \\ & \times (1_{(n-3)!\times(n-3)!} + f_3 M_3 + f_5 M_5 + f_3 f_3 M_3^3 + f_7 M_7 + f_3 f_5 M_3 M_5 + f_5 f_3 M_5 M_3 \\ & + f_9 M_9 + f_3 f_3 f_3 M_3^3 + f_5 f_5 M_5^2 + f_3 f_7 M_3 M_7 + f_7 f_3 M_7 M_3 \\ & + f_{11} M_{11} + f_3 f_3 f_5 M_3 M_3 M_5 + f_3 f_5 f_3 M_3 M_5 M_3 + f_5 f_3 f_3 M_5 M_3 M_3) + \mathcal{O}(\alpha'^{12}) \ . \end{aligned} \tag{3.14}$$

The coefficients P_{2k} of the commutative variables f_2^k build up a left-multiplicative matrix factor and can be cleanly disentangled from the odd-weight contributions involving f_{2i+1} and M_{2i+1}. Within the odd-weight sector, the democratic appearance of any non-commutative word in $f_{2i+1} M_{2i+1}$ with unit coefficient motivates the following generalization to arbitrary weight [14]:

$$\phi(F^{\mathfrak{m}}(s_{ij})) = \left(\sum_{k=0}^{\infty} f_2^k P_{2k} \right) \sum_{p=0}^{\infty} \sum_{\substack{i_1, i_2, \ldots, i_p \\ \in 2\mathbb{N}+1}} f_{i_1} f_{i_2} \ldots f_{i_p} M_{i_1} M_{i_2} \ldots M_{i_p} \ . \tag{3.15}$$

This all-order expression remains a conjecture beyond weights $\leq 21, 9, 7$ at multiplicity $n = 5, 6, 7$ where explicit checks have been performed in [33]. It is tempting to rewrite the right-multiplicative factor made of $f_{2i+1} M_{2i+1}$ as a formal geometric series,

$$\sum_{p=0}^{\infty} \sum_{\substack{i_1, i_2, \ldots, i_p \\ \in 2\mathbb{N}+1}} f_{i_1} f_{i_2} \ldots f_{i_p} M_{i_1} M_{i_2} \ldots M_{i_p} = \frac{1}{1 - (f_3 M_3 + f_5 M_5 + f_7 M_7 + \ldots)} \ , \tag{3.16}$$

whose equivalence to (3.15) and (3.1) relies on the expansion $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ under certain assumptions on the infinite series $x \equiv f_3 M_3 + f_5 M_5 + f_7 M_7 + \ldots$. Given the disappearance of the exotic rational prefactors $\frac{6}{25}$ or $-\frac{4}{35}$ in (3.5), the striking simplicity of (3.15) is manifested by the language based on words in f_2, f_{2i+1} where the coaction (3.10) takes a more intuitive form as compared to $\Delta\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}$. Hence, the understanding of the pattern in the α'-expansion can ultimately be attributed to the Hopf algebra structure of motivic MZVs.

Even though the coefficients of $f_{n_1+n_2+\ldots+n_p}$ in a given $\phi(\zeta^{\mathfrak{m}}_{n_1,n_2,\ldots,n_p})$ and thereby P_w, M_w depend on the choice of basis MZVs, the form of the end result (3.15) is universal. Explicit expressions for the matrices P_w, M_w at various weights w and multiplicities n are available for download at [31], they are associated with the MZV bases of [38]. At multiplicity $n=4$, they become scalars such that the vanishing of any commutator $[M_i, M_j]$ ensures the absence of depth ≥ 2 MZVs in the four-point amplitude. The closed-form expressions

$$M_{2i+1}\big|_{n=4} = -\frac{1}{2i+1}\big[s_{12}^{2i+1} + s_{23}^{2i+1} + s_{13}^{2i+1}\big] \ , \qquad P_{2k}\big|_{n=4} = \frac{\zeta_{2k}}{2k(\zeta_2)^k}\big[s_{12}^{2k} + s_{23}^{2k} - s_{13}^{2k}\big] \qquad (3.17)$$

with $s_{13} = -s_{12} - s_{23}$ can be inferred from the representation of $F^{(2)}$ in (2.20).

We emphasize that the complete information on $F^{\mathfrak{m}}(s_{ij})$ in contained in (3.15) since ϕ can be inverted to recover motivic MZVs from f_w. More importantly, only one matrix P_w, M_w along with f_w needs to be specified at each weight: The matrix-multiplicative pattern in (3.15) determines the coefficients of any other word in f_2 and f_{2i+1} of the same weight from matrices seen at lower weight. Given that the conjectural number [39] of linearly independent weight-w MZVs over $\mathbb{Q}$ grows with the order of $\left(\frac{4}{3}\right)^w$, this amounts to an enormous compression of information.

As firstly pointed out in [12], the form of the α'-expansion in (3.15) implies a simple expression for the motivic coaction,

$$\Delta F^{\mathfrak{m}}(s_{ij}) = F^{\mathfrak{m}}(s_{ij}) \otimes F^{\mathfrak{m}}(s_{ij})\big|_{\zeta_2^{\mathfrak{m}}=0} \ , \qquad (3.18)$$

where matrix multiplication is understood between the two sides of the tensor product. This resembles the coaction of the motivic Drinfeld associator $\Phi^{\mathfrak{m}}(e_0, e_1) \equiv \Phi(e_0, e_1)\big|_{\zeta_{n_1,\ldots,n_r} \to \zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}}$ [12]

$$\Delta\Phi^{\mathfrak{m}}(e_0, e_1) = \Phi^{\mathfrak{m}}(e_0, e_1) \overset{\triangleleft}{\otimes} \Phi^{\mathfrak{m}}(e_0, e_1)\big|_{\zeta_2^{\mathfrak{m}}=0} \ , \qquad (3.19)$$

where the operation $\triangleleft$ on top of the tensor product denotes the Ihara action among the words in e_0 and e_1 on the two sides. The expansion of the Drinfeld associator in a conjectural basis of MZVs over $\mathbb{Q}$ takes a form analogous to (3.15) where matrix multiplication among P_w and M_w is replaced by Ihara products among elementary words [12].

4 Further directions and open questions

In the previous sections, we have described the mathematical structure of tree-level amplitudes (1.1) among any number of massless open-superstring states. The string-corrections to the corresponding gauge-theory amplitudes are governed by the disk integrals in (1.2) whose α'-expansion exhibits elegant patterns of MZVs. As elaborated in section 2, the Drinfeld associator generates the dependence on dimensionless kinematic invariants $\alpha' k_i \cdot k_j$ in a recursive fashion, see in particular (2.1). Once the resulting MZVs are cast into their (conjectural) basis over $\mathbb{Q}$, their coefficients are related by matrix multiplication as displayed in (3.1). The systematics discussed in section 3 only become fully apparent if the MZVs are translated into a language based on non-commutative words. This dictionary is guided by the Hopf algebra structure, most notably by the coaction, and its mathematical validity relies on the use of motivic MZVs.

A couple of natural follow-up questions have already been addressed in the literature, so we shall conclude with a sketch of the subsequent developments before pointing out open problems.

4.1 The closed string at genus zero and single-valued MZVs

Tree-level scattering of closed strings is described by worldsheets of sphere topology. The integrations over vertex operator positions can be deformed in a way described in [9] such that closed-string tree amplitudes are composed from squares of open-string subamplitudes. This so-called "KLT-formula" [9] relies on the fact that the closed-string spectrum is contained in the tensor product of open-string excitations. At the massless level, for instance, closed-string excitations furnish a supersymmetry multiplet containing the graviton which arises from doubling gauge-boson supermultiplets in the open-string sector.

Once the $(n-3)!$-element basis of open-string subamplitudes [10, 11] is organized as in (3.3), the n-point closed-string tree amplitude $\mathcal{M}_n$ takes the form

$$\mathcal{M}_n(\alpha') = \sum_{\tau,\sigma,\rho,\pi \in S_{n-3}} \tilde{A}_{\rm YM}(1,\tau,n-1,n) F_\rho{}^\tau(s_{ij}) \mathcal{S}^{\rho;\pi}_{\alpha'}(s_{ij}) F_\pi{}^\sigma(s_{ij}) A_{\rm YM}(1,\sigma,n-1,n) \ , \qquad (4.1)$$

by the KLT-formula [9]. We use shorthands $\tilde{A}_{\rm YM}(1,\tau,n{-}1,n) \equiv \tilde{A}_{\rm YM}(1,\tau(2,\ldots,n{-}2),n{-}1,n)$ and $A_{\rm YM}(1,\sigma,n{-}1,n) \equiv A_{\rm YM}(1,\sigma(2,\ldots,n{-}2),n{-}1,n)$ for the two independent gauge-theory factors. The entries of the $(n-3)! \times (n-3)!$ matrix $\mathcal{S}^{\rho,\pi}_{\alpha'}(s_{ij})$ are degree $(n-3)!$ polynomials in $\sin(\pi s_{ij})$, see [9, 44, 45] for more details and various representations. Their α'-expansion is based on $\sin(\pi s_{ij}) = \pi s_{ij} \sum_{n=1}^{\infty} \frac{(-1)^n (\pi s_{ij})^{2n}}{(2n+1)!}$ and clearly interferes with the "even-zeta" sector represented by f_2 and P_{2k} in the ϕ-image (3.15) of disk integrals. In supergravity amplitudes obtained from the field-theory limit $\alpha' \to 0$ of (4.1), the sine functions are reduced to their argument, leaving behind

$$S^{\rho,\pi}_0(s_{ij}) \equiv \mathcal{S}^{\rho,\pi}_{\alpha'}(s_{ij})\big|_{\sin(\pi s_{ij}) \to \pi s_{ij}} \ . \tag{4.2}$$

It turns out that the properties of the matrices P_{2k} and $\mathcal{S}^{\rho,\pi}_{\alpha'}(s_{ij})$ lead to the striking cancellation of f_2 in the ϕ-image of the closed-string amplitude (4.1) [14],

$$\left(\sum_{k=0}^{\infty} f_2^k P_{2k}^t \right) \mathcal{S}_{\alpha'}(s_{ij}) \left(\sum_{l=0}^{\infty} f_2^l P_{2l} \right) = S_0(s_{ij}) \ , \tag{4.3}$$

which is tested to very high orders in α' (21, 9 and 7 at $n = 5, 6$ and 7) but remains conjectural beyond that. Another observational identity on the same footing concerns the matrices M_{2i+1} [14],

$$M^t_{2i+1} S_0(s_{ij}) = S_0(s_{ij}) M_{2i+1} \ , \tag{4.4}$$

which leads to additional cancellations among MZVs in the "odd-zeta" sector represented by the f_{2i+1} in the open-string α'-expansion. Taking both of (4.3) and (4.4) into account, the motivic version of the closed-string amplitude (4.1) defined in analogy to (3.13) can be simplified to [14]

$$\phi(\mathcal{M}^{\mathfrak{m}}_n(\alpha')) = \tilde{A}_{\rm YM} S_0(s_{ij}) \sum_{p=0}^{\infty} \sum_{\substack{i_1, i_2, \ldots, i_p \\ \in 2\mathbb{N}+1}} M_{i_1} M_{i_2} \ldots M_{i_p} \sum_{j=0}^{p} f_{i_1} f_{i_2} \ldots f_{i_j} \sqcup\!\sqcup\, f_{i_p} \ldots f_{i_{j+1}} A_{\rm YM} \ . \tag{4.5}$$

In all of (4.3) to (4.5), we have suppressed the S_{n-3}-"indices" present in (4.2) since the pattern of their summation is clear from the relative ordering of the matrices and vectors.

As pointed out in [15], the arrangement of the odd-weight variables f_{2i+1} in (4.5) implements the single-valued projection of MZVs [42, 46],

$$\text{sv}: \ f_{i_1} f_{i_2} \ldots f_{i_p} \to \sum_{j=0}^{p} f_{i_1} f_{i_2} \ldots f_{i_j} \sqcup\!\sqcup\, f_{i_p} \ldots f_{i_{j+1}} \ . \tag{4.6}$$

On these grounds, the α'-expansion in the representation (4.5) of the closed-string amplitude can be traced back to the single-valued version of the open-string amplitude (3.15) [15]

$$\mathcal{M}_n(\alpha') = \tilde{A}_{\rm YM} S_0(s_{ij}) {\rm sv}[A(\alpha')] \ , \tag{4.7}$$

where the $(n-3)!$ components of the vector $A(\alpha')$ on the right hand side are spelt out in (3.3). As detailed in [47], analogous statements hold for tree-level amplitudes of the heterotic string.

At the level of the associators, the single-valued projection (4.6) maps the Drinfeld associator to the Deligne associator [42] which therefore captures the structure of the closed-string amplitude [15]. It would be of central importance to find the closed-string counterpart of the recursive associator construction in section 2 [13]. The emergence of the single-valued projection in (4.5) and (4.7) could be rigorously proven from a direct derivation of the closed-string integrals from the Deligne associator and would not rely on the empirical properties of the matrices P_w and M_w in (4.3) and (4.4) which remain conjectural beyond certain orders.

4.2 The open string at genus one and elliptic MZVs

Apart from their implications for the closed string, the above results on open-string tree amplitudes call for a generalization to their quantum corrections and thereby to Riemann surfaces of higher genus. At the one-loop order of superstring perturbation theory, the worldsheet topologies relevant to open-string scattering are cylinder and Moebius-strip diagrams. For appropriate choice of the gauge group, these topologies conspire in a way to cancel infinities in the amplitudes considered in this section, and infinity cancellation in more general situations additionally involves the Klein-bottle topology [48]. Even though the cylindrical topology allows for insertions of vertex operators on both boundaries (see [49,50] for the implications on anomaly cancellations), we shall now report on recent studies [16] of the "planar" cylinder where the iterated integration is performed on a single boundary.

4.2.1 Definition and properties of elliptic MZVs

The mathematical framework for worldsheet integrals in planar one-loop amplitudes of the open superstring in known under the name of elliptic MZVs (eMZVs) [17, 18]. In the same way as MZVs can be defined as the expansion coefficients of the Drinfeld associator, see (2.9), eMZVs are defined [17] as the expansion coefficients of the elliptic Knizhnik-Zamolodchikov-Bernard (KZB) associator [18] which governs the regularized monodromy of the universal elliptic KZB equation.

This definition identifies eMZVs as iterated integrals on an elliptic curve $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ with $\mathrm{Im}\,(\tau) > 0$, in agreement with the approach via elliptic polylogarithms [51, 52]. The two homology cycles of the elliptic curve parametrized through the paths from $[0, 1]$ and $[0, \tau]$ give rise to two types of eMZVs, namely A-elliptic and B-elliptic MZVs. They descend from the two components $(A(\tau), B(\tau))$ of the elliptic KZB associator describing the monodromies of the elliptic KZB equation w.r.t. the paths $[0, 1]$ and $[0, \tau]$ and are related through the modular transformation $\tau \to -\frac{1}{\tau}$.

We will focus on A-elliptic MZVs associated with the homology cycle $[0, 1] \subset \mathbb{R}$ and, given that modular transformations restore the information on B-elliptic MZVs, refer to the former as eMZVs for simplicity. In this context, the definition of MZVs in (2.3) via iterated integrals generalizes to

$$\omega(n_1, n_2, \ldots, n_r; \tau) \equiv \int\limits_{0\leq z_1\leq z_2\leq \ldots \leq z_r\leq 1} \mathrm{d}z_1\, f^{(n_1)}(z_1, \tau)\, \mathrm{d}z_2\, f^{(n_2)}(z_2, \tau)\, \ldots\, \mathrm{d}z_r\, f^{(n_r)}(z_r, \tau) \tag{4.8}$$

with $n_j \in \mathbb{N}_0$ for $j = 1, 2, \ldots, r$. Instead of a two-letter alphabet $\{\frac{\mathrm{d}z}{z}, \frac{\mathrm{d}z}{1-z}\}$ of differential forms seen at genus zero, eMZVs in (4.8) exhibit an infinity of doubly-periodic functions $f^{(n)}$ which can be defined from their generating series

$$\exp\Big(2\pi i\alpha\frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}\Big)\frac{\theta'(0,\tau)\theta(z+\alpha,\tau)}{\theta(z,\tau)\theta(\alpha,\tau)} = \sum_{n=0}^{\infty}\alpha^{n-1}f^{(n)}(z,\tau)\ , \tag{4.9}$$

for instance $f^{(0)}(z, \tau) = 1$ and $f^{(1)}(z, \tau) = \frac{\partial}{\partial z}\ln\theta(z, \tau) + 2\pi i\frac{\mathrm{Im}\,(z)}{\mathrm{Im}\,(\tau)}$. The non-negative integers r and $w = n_1 + n_2 + \ldots + n_r$ in (4.8) are referred to as the length and the weight of an eMZV. The tick along with $\theta'(0, \tau)$ in (4.9) denotes a derivative of the odd Jacobi θ function w.r.t. its first argument z. Performing the integrals in the definition of eMZVs (4.8) yields a Fourier series in $q \equiv e^{2\pi i\tau}$ whose coefficients are MZVs along with integer powers of $2\pi i$ [17, 18].

By their definition (4.8) as iterated integrals, eMZVs satisfy shuffle relations

$$\omega(n_1, n_2, \ldots, n_r; \tau)\omega(k_1, k_2, \ldots, k_s; \tau) = \omega\big((n_1, n_2, \ldots, n_r) \sqcup\!\sqcup (k_1, k_2, \ldots, k_s); \tau\big)\ , \tag{4.10}$$

and the parity properties $f^{(n)}(-z, \tau) = (-1)^n f^{(n)}(z, \tau)$ following from $\theta(-z, \tau) = -\theta(z, \tau)$ and (4.9) imply the reflection identities

$$\omega(n_1, n_2, \ldots, n_{r-1}, n_r; \tau) = (-1)^{n_1+n_2+\ldots+n_r}\omega(n_r, n_{r-1}, \ldots, n_2, n_1; \tau)\,. \tag{4.11}$$

The combination of (4.10) and (4.11) is particularly constraining if the length r and the weight w

are both even or odd, i.e. if $r+w$ is even. In these cases, shuffle- and reflection identities can be used to express any such eMZV in terms of products of lower-length eMZVs [53].

As a higher-genus analogue of partial-fraction relations $\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$ among the genus-zero forms in MZVs (2.3), the doubly-periodic functions $f^{(n)}$ obey Fay relations [16, 52]

$$\begin{aligned} f^{(n_1)}(z-x,\tau)f^{(n_2)}(z,\tau) &= -(-1)^{n_1} f^{(n_1+n_2)}(x,\tau) + \sum_{j=0}^{n_2} \binom{n_1-1+j}{j} f^{(n_2-j)}(x,\tau) f^{(n_1+j)}(z-x,\tau) \\ &\quad + \sum_{j=0}^{n_1} \binom{n_2-1+j}{j} (-1)^{n_1+j} f^{(n_1-j)}(x,\tau) f^{(n_2+j)}(z,\tau) \ , \end{aligned} \tag{4.12}$$

which play a central rôle in deriving the subsequent α'-expansion of open-string one-loop amplitudes. Together with the shuffle- and reflection identities (4.10) and (4.11), the Fay relations were observed to generate all identities between eMZVs across a wide range of weights and lengths [53].

4.2.2 Elliptic MZVs in open-string amplitudes

The simplest non-vanishing one-loop amplitude of the open superstring involves four external massless states [8]. In the aforementioned planar cylinder topology, the four-point amplitude

$$A^{\text{1-loop}}(1,2,3,4;\alpha') = s_{12}s_{23}A_{\text{YM}}(1,2,3,4) \int_0^\infty \mathrm{d}t \ I_{1234}(\tau = it, s_{ij}) \tag{4.13}$$

is governed by the following iterated integral,

$$I_{1234}(\tau, s_{ij}) \equiv \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \ \prod_{i<j}^{4} e^{s_{ij} G(z_i - z_j, \tau)} \ , \tag{4.14}$$

with $z_1 = 0$ and $\mathrm{Re}\,(\tau) = 0$. The Mandelstam variables s_{ij} are defined in (1.3), and the bosonic Green function $G(z_i - z_j, \tau)$ satisfies

$$\frac{\partial}{\partial z} G(z,\tau) = f^{(1)}(z,\tau) \ , \qquad G(z,\tau) = \int_0^z \mathrm{d}x \, f^{(1)}(x,\tau) \ , \tag{4.15}$$

reflecting a regularization prescription for its zero mode that amounts to $G(0,\tau) \to 0$. Like this, the integral in (4.14) can be related to eMZVs in (4.8) and expanded at fixed values of τ [16],

$$\begin{aligned} I_{1234}(\tau, s_{ij}) &= \omega(0,0,0;\tau) \, - \, 2\omega(0,1,0,0;\tau)\,(s_{12}+s_{23}) \, + \, 2\omega(0,1,1,0,0;\tau)\left(s_{12}^2 + s_{23}^2\right) \\ &- 2\omega(0,1,0,1,0;\tau)\, s_{12}s_{23} \, + \, \beta_5(\tau)\,(s_{12}^3 + 2s_{12}s_{23}(s_{12}+s_{23}) + s_{23}^3) \, + \, \beta_{2,3}(\tau)\, s_{12}s_{23}(s_{12}+s_{23}) \, + \, \mathcal{O}(\alpha'^4) \ , \end{aligned} \tag{4.16}$$

with shorthands

$$\begin{aligned}\beta_5(\tau) &= \frac{4}{3}\big[\omega(0,0,1,0,0,2;\tau)+\omega(0,1,1,0,1,0;\tau)-\omega(2,0,1,0,0,0;\tau)-\zeta_2\omega(0,1,0,0;\tau)\big]\\ \beta_{2,3}(\tau) &= \frac{\zeta_3}{12}+\frac{8\zeta_2}{3}\,\omega(0,1,0,0;\tau)-\frac{5}{18}\,\omega(0,3,0,0;\tau)\ . \end{aligned} \tag{4.17}$$

At the third order in α', the Fay identities (4.12) are crucial to express the iterated integrals over three powers of the Green function (4.15) in terms of eMZVs. In an equivalent parametrization of the cylinder boundary via $z \in [0,\tau]$ instead of $z \in [0,1]$ as chosen in (4.14), the A-elliptic MZVs in (4.16) are traded for their B-elliptic analogues. In contrast to A-elliptic MZVs, however, B-elliptic MZVs are not periodic w.r.t. $\tau \to \tau{+}1$ and do not have Fourier expansion such as

$$\omega(0,1,0,0;\tau) = \frac{\zeta_3}{4\pi^2} + \frac{3}{2\pi^2}\sum_{m,n=1}^{\infty}\frac{1}{m^3}\,q^{mn} \tag{4.18}$$

for the A-elliptic MZV along with first α'-correction in (4.16). The Fourier expansion of the cylinder integral admitted by the parametrization in (4.14) has been exploited to check [16] that (4.16) reproduces the expected tadpole divergence [54]. The latter arises from the integration region $t \to i\infty$ in (4.13) and eventually cancels upon combination with the Moebius-strip diagram [48].

The polarization-dependence of the four-point amplitude (4.13) is represented by $A_{\rm YM}(1,2,3,4)$ and thereby follows the organization principle (1.1) of tree-level n-point amplitudes in terms of an $(n-3)!$-basis of subamplitudes $A_{\rm YM}(\ldots)$ [10, 11, 35]. Similarly, the five-point one-loop amplitude can be written as [16, 55]

$$A^{\text{1-loop}}(1,2,3,4,5;\alpha') = \int_0^\infty \mathrm{d}t \sum_{\sigma\in S_2} I_{1\sigma(23)45}(\tau=it,s_{ij})A_{\rm YM}(1,\sigma(2,3),4,5)\ , \tag{4.19}$$

see section 5.1 of [16] for more details on the integrals I_{12345} and I_{13245}. At higher multiplicity $n \geq 6$, a gauge invariant sector of open-string one-loop amplitudes has been reduced to field-theory subamplitudes as well [55]. However, the cancellation mechanism of the hexagon anomaly [49, 50] requires additional kinematic structures in $(n \geq 6)$-point amplitudes[6], so it remains an open problem to identify a suitable generalization of gauge-theory tree amplitudes to carry the polarization dependence of the string amplitude.

[6] In the pure spinor framework [19], kinematic building blocks suitable to describe the anomaly sector have been constructed in [56], see [57] for their appearance in the integrand of ten-dimensional field-theory amplitudes.

4.2.3 Bases of elliptic MZVs over $\mathbb{Q}$

Starting from the third subleading order in α', the increasing length and complexity of the eMZV-coefficients (4.17) calls for a systematic study of relations among eMZVs over $\mathbb{Q}$ and guiding principles to select a suitable basis. This has been done in [53], also see [58] for a particularly elaborate treatment of the length-two case. The number of independent eMZVs at given weight and length is bounded by their differential equation

$$\begin{aligned}
2\pi i \frac{\mathrm{d}}{\mathrm{d}\tau}\omega(n_1,\ldots,n_r;\tau) &= n_1 \mathrm{G}_{n_1+1}(\tau)\omega(n_2,\ldots,n_r;\tau) - n_r \mathrm{G}_{n_r+1}(\tau)\omega(n_1,\ldots,n_{r-1};\tau) \\
&+ \sum_{i=2}^{r} \Big\{ (-1)^{n_i}(n_{i-1}+n_i)\mathrm{G}_{n_{i-1}+n_i+1}(\tau)\omega(n_1,\ldots,n_{i-2},0,n_{i+1},\ldots,n_r;\tau) \\
&- \sum_{k=0}^{n_{i-1}+1}(n_{i-1}-k)\binom{n_i+k-1}{k}\mathrm{G}_{n_{i-1}-k+1}(\tau)\omega(n_1,\ldots,n_{i-2},k+n_i,n_{i+1},\ldots,n_r;\tau) \\
&+ \sum_{k=0}^{n_i+1}(n_i-k)\binom{n_{i-1}+k-1}{k}\mathrm{G}_{n_i-k+1}(\tau)\omega(n_1,\ldots,n_{i-2},k+n_{i-1},n_{i+1},\ldots,n_r;\tau)\Big\}
\end{aligned} \tag{4.20}$$

with $\mathrm{G}_n(\tau)$ denoting holomorphic Eisenstein series

$$\mathrm{G}_n(\tau) = \begin{cases} \displaystyle\sum_{\substack{k,m\in\mathbb{Z}\\(k,m)\neq(0,0)}} \frac{1}{(k+\tau m)^n} & : \ n>0 \\ -1 & : \ n=0 \ . \end{cases} \tag{4.21}$$

As a consequence of (4.20), eMZVs can be expressed in terms of iterated integrals over Eisenstein series, special cases of iterated Shimura integrals [59, 60]. In this picture, the iterated integration is carried out over the argument τ, and the counting of (shuffle-independent) iterated Eisenstein integrals sets an upper bound on the numbers of independent eMZVs.

On top of that, selection rules on the admissible Eisenstein integrals within eMZVs are encoded in an algebra of derivations [61–64] which appear in the differential equation of the elliptic KZB associator [17, 18], the generating series of eMZVs. In view of the central rôle of the Drinfeld associator for tree-level amplitudes seen in section 2, the elliptic associator is expected to carry essential information on one-loop open-string amplitudes including the α'-expansion (4.16).

A careful bookkeeping of eMZV relations within the above framework leads to the numbers $N(r,w)$ of indecomposable eMZVs[7] of length r and weight w as shown in table 1 [53]. The data

[7]A set of indecomposable eMZVs of weight w and length r is a minimal set of eMZVs such that any other eMZV of the same weight and length can be expressed as a linear combination of elements from this set as well as

in the table is compatible with the all-weight formulæ [53]

$$N(2,w) = 1 \ , \qquad N(3,w) = \left\lceil \frac{1}{6} w \right\rceil \ , \qquad N(4,w) = \left\lfloor \frac{1}{2} + \frac{1}{48}(w+5)^2 \right\rfloor \ , \tag{4.22}$$

which only hold for odd values of $r+w$ and remain conjectural at $r=4$.

r \ w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	1		1		1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		2		3		3		3		4		4	
4	1		1		2		3		4		5		7		8		10		12		14		16
5		1		2		4		6		9		13		17		23		30		37		47	
6	1		2		4		8		13		22		31		45		?		?		?		?
7		1		4		8		16		29		48		?		?		?		?		?	

Table 1: Numbers $N(r,w)$ of indecomposable eMZVs at length r and weight w.

Across a variety of lengths and weights, the decomposition of eMZVs in terms of such bases can be downloaded from [65], this website also contains new relations in the derivation algebra.

4.3 The closed string at higher genus

Closed-string amplitudes at one-loop originate from a worldsheet of torus topology. Again, the simplest non-vanishing superstring amplitude involves four massless external states [8], and the study of its α'-expansion has a rich history as well as strong motivation from S-duality of type-IIB superstring theory [66–68]. The α'-dependence stems from the worldsheet integral in the second line of

$$\begin{aligned} \mathcal{M}_4^{\text{1-loop}}(\alpha') = s_{12}^2 s_{23}^2 A_{\rm YM}(1,2,3,4) \tilde{A}_{\rm YM}(1,2,3,4) \\ \times \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,(\tau))^5} \int_{(\mathcal{T}_\tau)^3} \mathrm{d}^2 z_2 \, \mathrm{d}^2 z_3 \, \mathrm{d}^2 z_4 \prod_{i<j}^4 e^{s_{ij} G(z_i - z_j, \tau)} \ , \end{aligned} \tag{4.23}$$

analogous to (4.14) for the open string. The integration domain $\mathcal{T}_\tau$ is specified by the complex parametrization of the torus through a parallelogram with corners $0, 1, \tau+1, \tau$. The Green function in the exponent is defined in (4.15) and ensures modular invariance of the τ-integrand in (4.23) with $\mathcal{F}$ denoting the fundamental domain.

products of eMZVs with strictly positive weights and eMZVs of lengths smaller than r or weight lower than w. The coefficients are understood to comprise MZVs (including rational numbers) and integer powers of $2\pi i$.

The integration over τ leads to branch cuts in the dependence of the closed-string amplitude (4.23) on the Mandelstam variables s_{ij}, as required by unitarity. A procedure to reconcile the associated logarithmic dependence on s_{ij} with the naive Taylor expansion of the integral (4.23) has been described in [69], see [70] for recent updates. The discontinuity structure of the open-string one-loop amplitude follows the same principles and can be traced back to the integration over t in (4.13).

The systematic α'-expansion of the integrals arising from Taylor expanding $e^{s_{ij}G(z_i-z_j,\tau)}$ in (4.23) has been initiated in [71] and pursued in [69, 70]. In a representation of Green functions $G(z_i - z_j, \tau)$ as an edge between vertices i and j, intuitive graphical methods have been developed in these references, see [72,73] for an extension to the five-point one-loop amplitude. Since the zero mode of the Green function decouples from (4.23), only one-particle irreducible graphs contribute to the α'-expansion. The simplest class of such graphs have the topology of an n-gon, see figure 2, and the integration over z_2, z_3, z_4 in (4.23) gives rise to non-holomorphic Eisenstein series

$$E_n(\tau) \equiv \sum_{\substack{k,m\in\mathbb{Z}\\(k,m)\neq(0,0)}} \frac{(\mathrm{Im}\,(\tau))^n}{\pi^n\,|k+m\tau|^{2n}}\ , \qquad , n\in\mathbb{N}\ , \quad n\geq 2\ . \tag{4.24}$$

Beyond that, an infinite family of modular invariants has been classified and investigated in [70] (also see [74]), starting with the function

$$C_{2,1,1}(\tau) \equiv \sum_{\substack{k_1,k_2,m_1,m_2\in\mathbb{Z}\\(k_1,m_1),(k_2,m_2)\neq(0,0)\\(k_1+k_2,m_1+m_2)\neq(0,0)}} \frac{(\mathrm{Im}\,(\tau))^4}{\pi^4\,|k_1+m_1\tau|^2\,|k_2+m_2\tau|^2\,|k_1+k_2+(m_1+m_2)\tau|^4} \tag{4.25}$$

associated with the two-loop graph depicted in figure 2.

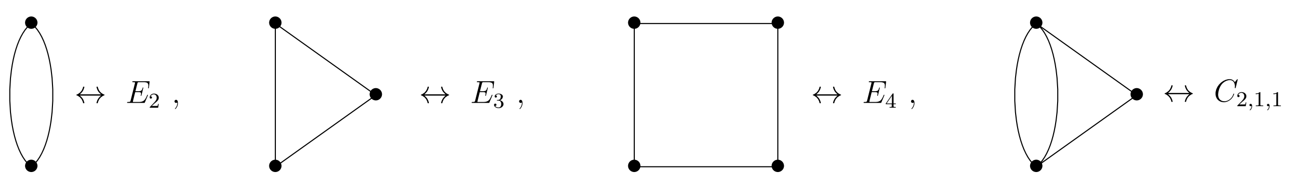

Figure 2: Graphical organization of several sample contributions to (4.23): Vertices represent the punctures z_i, $i = 1, 2, 3, 4$ and edges between the vertices for z_i and z_i are associated with a factor of $G(z_i - z_j, \tau)$. The integrals over z_2, z_3, z_4 become elementary in a Fourier expansion of the Green functions and yield the modular invariant lattice sums in (4.24) and (4.25).

The central rôle played by Laplace eigenvalue equations in the discussions of [70] such as

$$(\Delta - n(n-1))E_n(\tau) = 0 \ , \qquad (\Delta - 2)C_{2,1,1}(\tau) = 9E_4(\tau) - E_2^2(\tau) \tag{4.26}$$

with $\Delta = 4(\mathrm{Im}\,(\tau))^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar\tau}$ bears similarities to the methods of [17, 18, 53] to compute eMZVs from their differential equation in τ. It appears promising to investigate the parallels in the expansion of the open-string integral (4.14) and its closed-string counterpart (4.23) and to exploit cross-fertilizations of the methods used. Ultimately, it is tempting to hope for a one-loop generalization of the single-valued projection which was seen in (4.5) and (4.7) to map tree-level amplitudes of the open string to those of the closed string [15].

Certainly, the above structures deserve an investigation on higher-genus surfaces on the long run. While higher-genus generalizations of eMZVs have not yet appeared in the literature, the α'-expansion of the two-loop closed-string amplitude has been pushed beyond the leading order [75, 76] and led to a connection with mathematics literature on the so-called Zhang-Kawazumi invariant [77, 78]

$$\varphi(\Omega) \equiv \int_{\Sigma^2} \mathcal{G}(z,w,\Omega) \sum_{\substack{I,J,K,L\\=1,2}} \big[2(\mathrm{Im}\,\Omega)^{-1}_{IL}(\mathrm{Im}\,\Omega)^{-1}_{JK} - (\mathrm{Im}\,\Omega)^{-1}_{IJ}(\mathrm{Im}\,\Omega)^{-1}_{KL}\big]\omega_I(z)\overline{\omega_J(z)}\omega_K(w)\overline{\omega_L(w)} \ . \tag{4.27}$$

The arguments z, w of the genus-two Green function $\mathcal{G}(z,w,\Omega)$ are integrated over a genus-two Riemann surface with 2×2 period matrix Ω, and $\{\omega_I(z) : \ I = 1,2\}$ is a canonically normalized basis of holomorphic one forms. The Zhang-Kawazumi invariant φ in (4.27) can be viewed as the simplest two-loop analogue of the non-holomorphic Eisenstein series (4.24) and the modular invariants for more involved graph topologies in the one-loop α'-expansion.

It is not unlikely that string-theory questions at higher order in loops and α' encourage and even inspire the development of new mathematical structures.

Acknowledgments

I am very grateful to Johannes Broedel, Carlos Mafra, Nils Matthes, Stephan Stieberger and Tomohide Terasoma for collaboration on the projects on which this article is based. Moreover, I would like to thank Johannes Broedel, Nils Matthes and Federico Zerbini for valuable comments on the draft. I am indebted to the organizers of the conference "Numbers and Physics" in Madrid in September 2014 which strongly shaped the research directions leading to [16, 53] and possibly further results. I also acknowledge financial support by the European Research Council Advanced Grant No. 247252 of Michael Green.

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