One-loop string scattering amplitudes as iterated Eisenstein integrals

Johannes Broedel and Oliver Schlotterer

Abstract In these proceedings we review and expand on the recent appearance of iterated integrals on an elliptic curve in string perturbation theory. We represent the low-energy expansion of one-loop open-string amplitudes at multiplicity four and five as iterated integrals over holomorphic Eisenstein series. The framework of elliptic multiple zeta values serves as a link between the punctured Riemann surfaces encoding string interactions and the iterated Eisenstein integrals in the final results. In the five-point setup, the treatment of kinematic poles is discussed explicitly.

1 Introduction

Open-string scattering amplitudes at the one-loop level have proven to be a valuable laboratory for the application of techniques related to iterated elliptic integrals and elliptic multiple zeta values. Although elliptic curves and the classical elliptic integrals are one of the best-studied topics of $18^{\text{th}}/19^{\text{th}}$-century mathematics, iterated integrals on elliptic curves and their associated special values are still a prominent topic in the recent mathematics literature, see for instance refs. [1–3].

In high-energy physics, several integrals related to various scattering amplitudes in QCD have been solved using methods and techniques inherent to the elliptic curve. The concept of iterated integrals on an elliptic curve, however, made a first

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appearance in physics via one-loop scattering amplitudes in open-superstring theory in [4]. Since then, several refinements and extensions of the techniques have been put forward from different perspectives, see for examples refs. [5–10].

Moreover, first connections between the open-string setup of iterated integrals and non-holomorphic modular invariants encountered in closed-string amplitudes have been investigated in ref. [11]. The modular invariants in closed-string calculations are formulated in the framework of modular graph functions [12, 13], where tremendous progress in understanding their multiloop systematics has been made during the last couple of months [14, 15].

The low-energy expansion of one-loop scattering amplitudes in open-superstring theory gives rise to iterated elliptic integrals evaluated at special points: those functions of the modular parameter τ of the elliptic curve are called elliptic multiple zeta values and come in a twisted and an untwisted version. Both, untwisted and twisted elliptic multiple zeta values, however, allow for an alternative representation in terms of iterated integrals over the modular parameter τ: iterated Eisenstein integrals.

In these proceedings we are extending earlier results in two directions: we present low-energy expansions for the planar and non-planar five-point amplitudes, and we cast the four- and five-point expressions in the language of iterated Eisenstein integrals.

The current proceedings are structured as follows: in section 2 we provide background information and define the mathematical setting for the calculation of one-loop open-string amplitudes at various multiplicities. We classify the occurring integrals and state the integral contributions to be evaluated at the four- and five-point level. In section 3 a short introduction to twisted and untwisted elliptic multiple zeta values is provided. We relate these special values to iterated integrals over different flavors of Eisenstein series. This representation allows to infer relations between different twisted and untwisted elliptic multiple zeta values, which paves the way towards a canonical representation. Accordingly, in sections 4 and 5 we present and discuss the results of the four- and five-point integrals from section 2 and represent them in terms of conventional elliptic multiple zeta values as well as iterated integrals over Eisenstein series.

2 One-loop open-string amplitudes, planar and non-planar

2.1 General setup, planar and non-planar

Scattering amplitudes in string theories are derived from punctured Riemann surfaces called worldsheets whose genus corresponds to the loop order in perturbation theory. In these proceedings we are going to consider the one-loop order exclusively, where the relevant topology for closed strings is a torus, and open-string amplitudes receive contributions from worldsheets of cylinder- and Mœbius-strip topologies. In all cases, the punctures correspond to the insertion of external states on the worldsheet via vertex operators; those are conformal primary fields that carry the information on the external momenta and polarizations. For open strings, the

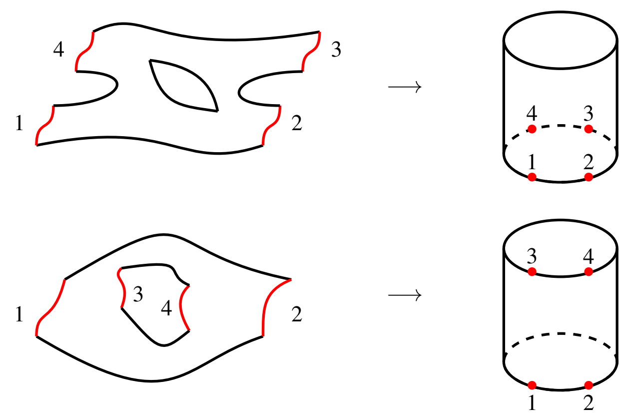

Fig. 1 The worldsheets for one-loop scattering of open strings include the topology of a cylinder. Conformal invariance on the worldsheet can be used to map external states to punctures on the cylinder boundaries. If vertex operators are inserted on one boundary only, the situation is referred to as the *planar cylinder* whereas the second topology is called the *non-planar cylinder*.

vertex operators are inserted on the worldsheet boundaries, see figure 1. Moreover, each external open-string state carries additional degrees of freedom encoded in Lie-algebra generators t^a, called Chan–Paton factors. They enter scattering amplitudes in the form of traces, where the ordering of the generators reflects the distribution of vertex operators over the boundaries [16]. We will only consider massless vibration modes of the open superstring as an external state, i.e. one-loop scattering of gauge bosons and their superpartners. Accordingly, the Chan–Paton degrees of freedom of the external states are often referred to as color.

Having a single boundary only, the Mœbius strip can only contribute single traces to the n-point amplitude

$$M^n_{\text{Moeb}} = -32 \sum_{\rho \in S_{n-1}} \text{Tr}(t^1 t^{\rho(2)} t^{\rho(3)} \dots t^{\rho(n)}) A_{\text{Moeb}}(1, \rho(2), \rho(3), \dots, \rho(n)) \,, \quad (1)$$

while the two boundary components of the cylinder admit double traces in the color decomposition. Accordingly, for a four-point amplitude the planar and non-planar cylinder contributions read

$$
\begin{aligned}
M^4_{\rm cyl} = \sum_{\rho\in S_3} \big\{ & N\,{\rm Tr}(t^1 t^{\rho(2)} t^{\rho(3)} t^{\rho(4)}) A_{\rm cyl}(1,\rho(2),\rho(3),\rho(4)) \\
&+ {\rm Tr}(t^1 t^{\rho(2)}){\rm Tr}(t^{\rho(3)} t^{\rho(4)}) A_{\rm cyl}(1,\rho(2)|\rho(3),\rho(4)) \big\} \\
&+ \big\{ {\rm Tr}(t^1){\rm Tr}(t^2 t^3 t^4) A_{\rm cyl}(1|2,3,4) + (1\leftrightarrow 2,3,4) \big\}\,.
\end{aligned} \tag{2}
$$

At higher multiplicity n, the analogous double-trace expressions in

$$
M^n_{\rm cyl} = N \sum_{\rho\in S_{n-1}} {\rm Tr}(t^1 t^{\rho(2)} \ldots t^{\rho(n)}) A_{\rm cyl}(1,\rho(2),\ldots,\rho(n)) + \text{double traces}\,, \tag{3}
$$

comprise all partitions of the external states over the two boundaries along with all cyclically inequivalent arrangements. For instance, the double-trace sector of the five-point amplitude features permutations of ${\rm Tr}(t^1t^2){\rm Tr}(t^3t^4t^5)A_{\rm cyl}(1,2|3,4,5)$ and ${\rm Tr}(t^1){\rm Tr}(t^2t^3t^4t^5)A_{\rm cyl}(1|2,3,4,5)$, with an obvious generalization to higher multiplicity.

The number N of colors in the single-trace sector of eqs. (2) and (3) arises from the trace over the identity matrix corresponding to the empty boundary component. The color-ordered amplitudes $A_{\rm Moeb}$ and $A_{\rm cyl}$ in eqs. (1) and (3) are determined by integrating a correlation function of vertex operators over the punctures such that their cyclic ordering on each boundary component matches the accompanying color traces [16]. In the parametrization of the cylinder as half of a torus with purely imaginary modular parameter $\tau = it$, $t\in\mathbb{R}$, see figure 2, the integration domains for the punctures are of the form

$$
\begin{aligned}
D(1,2,\ldots,j|j{+}1,\ldots,n) = \{ & z_i \in \mathbb{C},\ {\rm Im}\, z_{1,2,\ldots,j} = 0,\ {\rm Im}\, z_{j+1,\ldots,n} = \tfrac{t}{2}\,, \\
& 0 \leq {\rm Re}\, z_1 < {\rm Re}\, z_2 < \ldots < {\rm Re}\, z_j < 1,\ 0 \leq {\rm Re}\, z_{j+1} < \ldots < {\rm Re}\, z_n < 1 \}\,.
\end{aligned} \tag{4}
$$

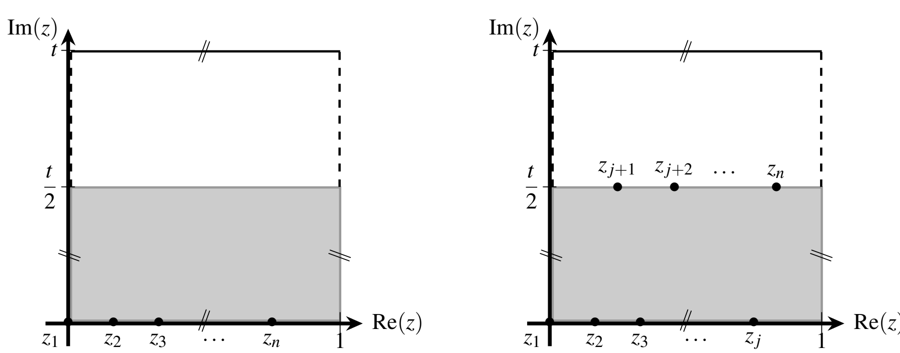

Fig. 2 In the boundary parametrization eq. (4), worldsheets of cylinder topology are mapped to the shaded regions in the left (right) panel for the planar (non-planar) case. These regions cover half of a torus with modular parameter $\tau = it$ and identifications of edges marked by ⋍ and //, respectively. The Mœbius-strip topology is not drawn here as its contributions to the amplitude can be inferred from the planar cylinder [17], cf. eqs. (6) and (8).

In particular, eq. (4) refers to the non-planar amplitude $A_{\rm cyl}(1,2,\ldots,j|j{+}1,\ldots,n)$ along with the double trace $\mathrm{Tr}(t^1t^2\ldots t^j)\mathrm{Tr}(t^{j+1}\ldots t^n)$ with $j=1,2,\ldots,n{-}1$. We will also write $D(1,2,\ldots,n)=D(1,2,\ldots,n|\emptyset)$ for the integration domain of the planar cylinder amplitude $A_{\rm cyl}(1,2,\ldots,n)$ in eq. (3).

The correlation functions in the integrand will be denoted by $\mathcal{K}_n$. They depend on the punctures z_j, the modular parameter τ as well as the external polarizations and momenta of the gauge supermultiplet. For the cylinder topology, the integration domain for modular parameters $\tau=it$ is $t\in\mathbb{R}_+$ or

$$q=e^{2\pi i\tau}=e^{-2\pi t}\,,\qquad q\in(0,1)\,. \tag{5}$$

Then, the expression for color-ordered cylinder amplitudes reads

$$A_{\rm cyl}(1,2,\ldots,j|j{+}1,\ldots,n)=\int_0^1\frac{\mathrm{d}q}{q}\int\limits_{D(1,2,\ldots,j|j+1,\ldots,n)}\mathrm{d}z_1\,\mathrm{d}z_2\,\ldots\,\mathrm{d}z_n\,\delta(z_1)\,\mathcal{K}_n\,, \tag{6}$$

where translation invariance on a genus-one surface has been used to fix $z_1=0$ through a delta-function insertion. We will also express the punctures in eq. (4) in terms of real variables $x_i\in(0,1)$ and parametrize $D(1,2,\ldots,j|j{+}1,\ldots,n)$ via

$$z_i=\left\{\begin{smallmatrix} x_i & : \, i=1,2,\ldots,j \\ \frac{\tau}{2}+x_i & : \, i=j+1,\ldots,n\end{smallmatrix}\right.\,,\quad 0\leq x_1<x_2<\ldots<x_j<1\,,\quad 0\leq x_{j+1}<\ldots<x_n<1\,. \tag{7}$$

For single-trace amplitudes in eq. (6) with $j=n$, the integration over q introduces endpoint divergences as $q\to 0$. The latter cancel against the divergent contributions from the Mœbius strip in eq. (1)

$$A_{\rm Moeb}(1,2,\ldots,n)=\int_0^{-1}\frac{\mathrm{d}q}{q}\int\limits_{D(1,2,\ldots,n)}\mathrm{d}z_1\,\mathrm{d}z_2\,\ldots\,\mathrm{d}z_n\,\delta(z_1)\,\mathcal{K}_n \tag{8}$$

if $N=32$, i.e. if the gauge group[1] is taken to be $SO(32)$ [17]. The change of variables leading to the range $q\in(-1,0)$ in eq. (8) can also be found in the reference.

In this work, we will be interested in the low-energy expansion of the integrals over the cylinder punctures in eq. (6) at fixed value of q but unspecified choice of the gauge group. For instance, the integrals over $D(1,2,3|4)$ turn out to have an interesting mathematical structure, even though their coefficients $\sim\mathrm{Tr}(t^4)$ vanish for the physically preferable gauge group $SO(32)$. At the level of the integrand w.r.t. q, the Mœbius-strip results in eq. (8) can be inferred from the planar instance of eq. (6) by sending $q\to -q$ [17].

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[1] The choice of gauge group $SO(32)$ also ensures that the hexagon gauge anomaly in $(n\geq 6)$-point open-superstring amplitudes cancels [18, 19].

2.2 Four-point amplitudes

The four-point cylinder amplitude eq. (6) of massless open-superstring states is governed by the correlation function

$$\mathscr{K}_4 = s_{12}\, s_{23}\, A^{\text{tree}}_{\text{SYM}}(1,2,3,4) \prod_{i<j}^{4} \exp\left(\tfrac{1}{2} s_{ij} G(z_{ij},\tau)\right), \tag{9}$$

which has firstly been derived for external bosons in 1982 [20]. The exponentials of eq. (9) involve dimensionless Mandelstam variables s_{ij}

$$s_{ij} = 2\alpha' k_i \cdot k_j \tag{10}$$

with inverse string tension α'. Moreover, eq. (9) features the bosonic Green function on a genus-one worldsheet

$$G(z,\tau) = \log\left|\frac{\theta_1(z,\tau)}{\theta_1'(0,\tau)}\right|^2 - \frac{2\pi}{\operatorname{Im}\tau}(\operatorname{Im} z)^2 . \tag{11}$$

with $z_{ij} = z_i - z_j$ as its first argument, where θ_1 is the odd Jacobi function

$$\theta_1(z,\tau) = 2q^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1-q^n)(1-2q^n \cos(2\pi z) + q^{2n}) . \tag{12}$$

Finally, external polarizations enter eq. (9) through the color-ordered (super-)Yang–Mills tree-level amplitude $A^{\text{tree}}_{\text{SYM}}(1,2,3,4)$.

With respect to relabeling of the external legs, there are three inequivalent representatives for the planar and non-planar four-point amplitudes. Using eqs. (6) and (9), they can be written as

$$\begin{aligned}
A_{\text{cyl}}(1,2,3,4) &= s_{12}s_{23}A^{\text{tree}}_{\text{SYM}}(1,2,3,4) \int_0^1 \frac{\mathrm{d}q}{q}\, I_{1234}(s_{ij},q) \\
A_{\text{cyl}}(1,2,3|4) &= \frac{1}{2} s_{12}s_{23}A^{\text{tree}}_{\text{SYM}}(1,2,3,4) \int_0^1 \frac{\mathrm{d}q}{q}\, I_{123|4}(s_{ij},q) \\
A_{\text{cyl}}(1,2|3,4) &= \frac{1}{2} s_{12}s_{23}A^{\text{tree}}_{\text{SYM}}(1,2,3,4) \int_0^1 \frac{\mathrm{d}q}{q}\, I_{12|34}(s_{ij},q) ,
\end{aligned} \tag{13}$$

where the integrals over the positions of the punctures defined in eq. (7) read

$$\begin{aligned}
I_{1234}(s_{ij},q) &= \int_0^1 \mathrm{d}x_4 \int_0^{x_4} \mathrm{d}x_3 \int_0^{x_3} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1\, \delta(x_1) \exp\Big(\sum_{i<j}^{4} \frac{s_{ij}}{2} G(x_{ij})\Big) \\
I_{123|4}(s_{ij},q) &= \Big(\prod_{l=1}^{4} \int_0^1 \mathrm{d}x_l\Big)\, \delta(x_1) \exp\Big(\sum_{i<j}^{3} \frac{s_{ij}}{2} G(x_{ij}) + \sum_{j=1}^{3} \frac{s_{j4}}{2} G(\tfrac{\tau}{2} + x_{ij})\Big)
\end{aligned} \tag{14}$$

$$I_{12|34}(s_{ij},q) = \Big(\prod_{l=1}^{4}\int_0^1 \mathrm{d}x_l\Big)\,\delta(x_1)\exp\Big(\frac{s_{12}}{2}G(x_{12})+\frac{s_{34}}{2}G(x_{34})+\sum_{\substack{i=1,2\\ j=3,4}}\frac{s_{ij}}{2}G(\tfrac{\tau}{2}+x_{ij})\Big)\,.$$

Here and below, the dependence on τ in the Green functions is left implicit for ease of notation. The factors of $\frac{1}{2}$ in eq. (13) are introduced to obtain a more convenient description of the integration domain for the non-planar cases $I_{123|4}(s_{ij},q)$ and $I_{12|34}(s_{ij},q)$: The natural integration domains $0 \le x_1 < x_2 < x_3 < 1$ and $0 \le x_3 < x_4 < 1$ expected from $\mathrm{Tr}(t^1t^2t^3)$ and $\mathrm{Tr}(t^3t^4)$ can be rewritten to yield an independent integration of all the x_i over $(0,1)$ when taking the symmetry of the color factors or the integrands into account.

The integrals in eq. (14) are the central four-point quantities in these proceedings. In section 4, we are going to review and extend the results of refs. [4, 6] on their low-energy expansion around $\alpha'=0$, i.e. the Taylor expansion in the dimensionless Mandelstam invariants eq. (10). Note that momentum conservation and the choice of massless external states in eq. (9) with $k_j^2=0\ \forall\ j=1,2,3,4$ relate the four-point Mandelstam invariants

$$\sum_{j=1}^{4} k_j = 0 \quad\Rightarrow\quad s_{12}=s_{34}\,,\quad s_{14}=s_{23}\,,\quad s_{13}=s_{24}=-s_{12}-s_{23}\,. \tag{15}$$

Accordingly, the integrand in eq. (9) is unchanged if the Green function is shifted by $G(z,\tau)\to G(z,\tau)+f(\tau)$ as long as $f(\tau)$ does not depend on the position of the punctures.

2.3 Five-point amplitudes

The massless five-point correlator for the cylinder amplitude eq. (6) is given by[2] [23, 24]

$$\mathcal{K}_5 = \big[f_{23}^{(1)}s_{23}C_{1|23,4,5}+(23\leftrightarrow 24,25,34,35,45)\big]\prod_{i<j}^{5}\exp\big(\tfrac{1}{2}s_{ij}G(z_{ij})\big)\,, \tag{16}$$

where the Green function is defined in eq. (11) and we use the following shorthand for doubly-periodic functions of the punctures with a simple pole at $z_i - z_j \in \mathbb{Z}+\tau\mathbb{Z}$

$$f_{ij}^{(1)} = \partial_z \log\theta_1(z_{ij},\tau)+2\pi i\frac{\mathrm{Im}\,z_{ij}}{\mathrm{Im}\,\tau} = \partial_z G(z_{ij},\tau)\,. \tag{17}$$

The kinematic factors in eq. (16) obey symmetries $C_{1|23,4,5}=C_{1|23,5,4}=-C_{1|32,4,5}$ and can be expressed in terms of (super-)Yang–Mills tree-level amplitudes [24]

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[2] Earlier work on five- and higher-point correlation functions for one-loop open-superstring amplitudes includes refs. [21, 22].

$$C_{1|23,4,5} = s_{45}\big[s_{24}A^{\rm tree}_{\rm SYM}(1,3,2,4,5) - s_{34}A^{\rm tree}_{\rm SYM}(1,2,3,4,5)\big]\,. \tag{18}$$

The color decomposition of the five-point cylinder amplitude is a straightforward generalization of eq. (2), and we collectively denote the color-ordered amplitudes by $A_{\rm cyl}(\lambda)$ with $\lambda = 1,2,3,4,5$ in the planar and $\lambda = 1,2,3,4|5$ or $\lambda = 1,2,3|4,5$ in the non-planar sector. Then, one can combine eqs. (16) and (18) to bring all the cylinder contributions to the five-point amplitude into the form

$$A_{\rm cyl}(\lambda) = \int_0^1 \frac{{\rm d}q}{q}\,\big[I^{23}_\lambda(s_{ij},q)A^{\rm tree}_{\rm SYM}(1,2,3,4,5)+I^{32}_\lambda(s_{ij},q)A^{\rm tree}_{\rm SYM}(1,3,2,4,5)\big] \tag{19}$$

for some integrals $I^{23}_\lambda(s_{ij},q)$ and $I^{32}_\lambda(s_{ij},q)$ over the punctures whose domain $D(\lambda)$ is defined by eq. (4). The color-ordered (super-)Yang-Mills amplitudes obtained from relabelings of eq. (18) have been written in terms of a two-element basis of $A^{\rm tree}_{\rm SYM}(1,2,3,4,5)$ and $A^{\rm tree}_{\rm SYM}(1,3,2,4,5)$ using Bern–Carrasco–Johansson (BCJ) relations [25]. For planar choices of λ, for example, both $I^{23}_\lambda(s_{ij},q)$ and $I^{32}_\lambda(s_{ij},q)$ can be reduced to the following permutation-inequivalent integrals

$$H^{12}_{12345}(s_{ij},q) = \int_0^1 {\rm d}x_5\Big(\prod_{l=1}^4\int_0^{x_{l+1}}{\rm d}x_l\Big)\,\delta(x_1)\,f^{(1)}_{12}\,\exp\Big(\sum_{i<j}^5\frac{s_{ij}}{2}G(x_{ij})\Big) \tag{20}$$

$$\widehat{H}^{13}_{12345}(s_{ij},q) = \int_0^1 {\rm d}x_5\Big(\prod_{l=1}^4\int_0^{x_{l+1}}{\rm d}x_l\Big)\,\delta(x_1)\,f^{(1)}_{13}\,\exp\Big(\sum_{i<j}^5\frac{s_{ij}}{2}G(x_{ij})\Big)\,. \tag{21}$$

The hat-notation in (21) and (23) below is used to distinguish integrals $\widehat{H}^{ij}_\lambda$ with a regular Taylor expansion around $s_{ij}{=}0$ from cases H^{ij}_λ with kinematic poles of the form s_{ij}^{-1}, see section 5.1. In the non-planar sector with $\lambda = 1,2,3|4,5$, on the other hand, $I^{23}_\lambda(s_{ij},q)$ and $I^{32}_\lambda(s_{ij},q)$ can be assembled from permutations of

$$H^{12}_{123|45}(s_{ij},q) = \Big(\prod_{l=3}^5\int_0^1{\rm d}x_l\Big)\int_0^{x_3}{\rm d}x_2\int_0^{x_2}{\rm d}x_1\,\delta(x_1)\,f^{(1)}_{12}\,\exp\Big(\sum_{i<j}^5\frac{s_{ij}}{2}G(\delta_{ij}\tfrac{\tau}{2}{+}x_{ij})\Big) \tag{22}$$

$$\widehat{H}^{14}_{123|45}(s_{ij},q) = \Big(\prod_{l=3}^5\int_0^1{\rm d}x_l\Big)\int_0^{x_3}{\rm d}x_2\int_0^{x_2}{\rm d}x_1\,\delta(x_1)\,f^{(1)}_{14}\,\exp\Big(\sum_{i<j}^5\frac{s_{ij}}{2}G(\delta_{ij}\tfrac{\tau}{2}{+}x_{ij})\Big)\,, \tag{23}$$

where $\delta_{12}=\delta_{13}=\delta_{23}=\delta_{45}=0$ and $\delta_{ij}=1$ if $i=1,2,3$ and $j=4,5$. The analogous non-planar integral with $f^{(1)}_{45}$ in the place of $f^{(1)}_{12}$ and $f^{(1)}_{14}$ vanishes, because the integration measure is symmetric in $4,5$ while $f^{(1)}_{45}=-f^{(1)}_{54}$,

$$\Big(\prod_{l=3}^5\int_0^1{\rm d}x_l\Big)\int_0^{x_3}{\rm d}x_2\int_0^{x_2}{\rm d}x_1\,\delta(x_1)\,f^{(1)}_{45}\,\exp\Big(\sum_{i<j}^5\frac{s_{ij}}{2}G(\delta_{ij}\tfrac{\tau}{2}+x_{ij})\Big) = 0\,. \tag{24}$$

Note that there are five independent Mandelstam variables for five massless particles, for example $s_{12}, s_{23}, s_{34}, s_{45}, s_{51}$,

$$\sum_{j=1}^{5} k_j = 0 \quad \Rightarrow \quad s_{13} = s_{45} - s_{12} - s_{23} \text{ and } \mathrm{cyc}(1,2,3,4,5)\,. \tag{25}$$

Similarly, the non-planar sector with $\lambda = 1,2,3,4|5$ admits three topologies of permutation-inequivalent integrals: with insertions $f_{12}^{(1)}, f_{13}^{(1)}$ and $f_{45}^{(1)}$ beyond the Koba–Nielsen-factor, respectively.

2.4 Higher-point amplitudes

Starting from six external states, the correlators $\mathscr{K}_n$ no longer boil down to tree-level amplitudes $A^{\rm tree}_{\rm SYM}(\ldots)$ in (super-)Yang–Mills theory. Instead, one finds a more general class of kinematic factors, see refs. [26, 27] for their precise form and the accompanying functions of the punctures at six points.

3 Mathematical tools/objects

Employing the form of the open-string one-loop propagator in eq. (9) and expanding the exponentials of the propagators in powers of α' (cf. eq. (10)), one finds all integrals in the previous section to boil down to iterated integrals on the elliptic curve. The integration kernels $f_{ij}^{(1)}$ in eq. (17) and their higher-weight generalizations are canonical differentials on the elliptic curve that can be generated by a non-holomorphic extension of the Eisenstein–Kronecker series [1, 28]

$$\Omega(z,\alpha,\tau) = \exp\Big(2\pi i\alpha \frac{\mathrm{Im}\, z}{\mathrm{Im}\, \tau}\Big) \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)} = \sum_{n=0}^{\infty} \alpha^{n-1} f^{(n)}(z,\tau)\,. \tag{26}$$

The expansion in the second equality yields doubly-periodic functions

$$f^{(n)}(z,\tau) = f^{(n)}(z+1,\tau) = f^{(n)}(z+\tau,\tau)\,, \qquad f^{(n)}(-z,\tau) = (-1)^n f^{(n)}(z,\tau)\,, \tag{27}$$

for example $f^{(0)} = 1$ and $f^{(1)}(z,\tau) = \partial_z \log\theta_1(z,\tau) + 2\pi i \frac{\mathrm{Im}\, z}{\mathrm{Im}\, \tau}$. Equation (17) arises from the shorthand $f_{ij}^{(n)} = f^{(n)}(z_i - z_j, \tau)$. The Fay relations of the Eisenstein–Kronecker series [1, 29]

$$\Omega(z_1,\alpha_1,\tau)\Omega(z_2,\alpha_2,\tau) = \Omega(z_1,\alpha_1+\alpha_2,\tau)\Omega(z_2-z_1,\alpha_2,\tau) + (z_1,\alpha_1 \leftrightarrow z_2,\alpha_2) \tag{28}$$

imply the following component relations when Laurent expanded in the bookkeeping variables α_i [4]:

$$f_{ij}^{(n)} f_{jl}^{(m)} = -f_{il}^{(m+n)} + \sum_{k=0}^{n} (-1)^k \binom{m-1+k}{k} f_{il}^{(n-k)} f_{jl}^{(m+k)} \\ + \sum_{k=0}^{m} (-1)^k \binom{n-1+k}{k} f_{il}^{(m-k)} f_{ij}^{(n+k)} \,. \tag{29}$$

As already noted for the Green function after eq. (14), all functions considered in these proceedings are functions of the modular parameter τ, which we will suppress here and below. Using the integration kernels $f^{(n)}(z)$ and the following definition of elliptic iterated integrals[3] with $\Gamma(\,;z) = 1$,

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_\ell \\ b_1 & b_2 & \dots & b_\ell \end{smallmatrix} ;z \right) = \int_0^z \mathrm{d}t\, f^{(n_1)}(t-b_1)\, \Gamma\left(\begin{smallmatrix} n_2 & \dots & n_\ell \\ b_2 & \dots & b_\ell \end{smallmatrix} ;t \right) , \qquad z \in [0,1] \,, \tag{30}$$

one can solve the integrals over the punctures z_j in one-loop open-superstring amplitudes order by order in α'. In particular, it will be explained in detail in section 4 how the mathematical tools of this section yield a recursive and algorithmic procedure to expand the four-point integrals eq. (14) to any desired order in α'.

Allowing for rational values s_i and r_i in the fundamental elliptic domain only, twists $b_i = s_i + r_i\tau$ with $r_i, s_i \in [0,1)$ lead to the notion of twisted elliptic multiple zeta values or teMZVs [6]:

$$\begin{aligned} \omega\left(\begin{smallmatrix} n_1, & n_2, & \dots, & n_\ell \\ b_1, & b_2, & \dots, & b_\ell \end{smallmatrix} \right) &= \int\limits_{0 \le z_i \le z_{i+1} \le 1} f^{(n_1)}(z_1-b_1)\mathrm{d}z_1\, f^{(n_2)}(z_2-b_2)\mathrm{d}z_2 \dots f^{(n_\ell)}(z_\ell-b_\ell)\mathrm{d}z_\ell \\ &= \quad \Gamma\left(\begin{smallmatrix} n_\ell & n_{\ell-1} & \dots & n_1 \\ b_\ell & b_{\ell-1} & \dots & b_1 \end{smallmatrix} ;1 \right) . \end{aligned} \tag{31}$$

If there are no twists, that is, $b_i = 0 \;\forall\; i$, one obtains untwisted elliptic multiple zeta values or eMZVs, for which a simplified notation is used [4, 5]:

$$\omega(n_1, n_2, \dots, n_\ell) = \Gamma\left(\begin{smallmatrix} n_\ell & \dots & n_2 & n_1 \\ 0 & \dots & 0 & 0 \end{smallmatrix} ;1 \right) = \Gamma(n_\ell, \dots, n_2, n_1; 1) \,. \tag{32}$$

For eMZVs and teMZVs defined in eqs. (31) and (32), the quantities $w = \sum_{i=1}^{\ell} n_i$, and the number ℓ of integrations in are referred to as *weight* and *length* of the elliptic iterated integral and the corresponding (t)eMZV, respectively.

In view of the simple pole of $f^{(1)}(z,\tau)$ at $z = 0,1$, eMZVs with entries $n_1 = 1$ or $n_\ell = 1$ suffer from endpoint divergences, whose regularization was discussed in ref. [4]. Similarly, a regularization scheme for the divergences caused by twists $b \in \mathbb{R}$ in eq. (31) can be found in ref. [6].

[3] The iterated integrals in eq. (30) are not homotopy invariant. Still, one can find a homotopy-invariant completion for each $\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_\ell \\ b_1 & b_2 & \dots & b_\ell \end{smallmatrix} ;z \right)$ from the generating series in ref. [1] (see also subsection 3.1 of ref. [4]).

3.1 Elliptic multiple zeta values in terms of iterated Eisenstein integrals

While teMZVs can be represented as iterated integrals over the positions z_i of vertex operators, the analytically favorable way is to convert them to iterated integrals in the modular parameters τ. The main reason is, that the integration kernels appearing in this setting are very well-known objects: holomorphic Eisenstein series for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of various levels M. For level 1, iterated τ-integrals over Eisenstein series do not satisfy any relations except for shuffle [30], hence, representing these (untwisted) eMZVs in terms of iterated Eisenstein integrals automatically exposes all their relations over the rational numbers. For levels $M>1$, however, the Eisenstein series are not independent, when evaluated at rational points of the lattice. These relations have been investigated and discussed in ref. [10] and allow to relate different iterated integrals, even between different levels M.

There does exist a straightforward method for converting iterated z-integrals underlying (t)eMZVs to iterated Eisenstein integrals $\mathscr{E}_0$ over Eisenstein series [2,5,6]: since the resulting "number" is still going to be a function of the modular parameter τ, one can conveniently take a derivative with respect to τ. Let us make this construction precise in the next paragraphs.

Given a teMZV of the form (31), let us take all of the twists b_i from a rational lattice $\Lambda_M=\left\{\frac{r}{M}+\tau\frac{s}{M}:\, r,s=0,1,2,\ldots,M-1\right\}$ within the elliptic curve characterized by an integer level $M\in\mathbb{N}$. The derivative in τ of the teMZV is most conveniently expressed in terms of functions[4]

$$h^{(n)}(b_i,\tau)=(n-1)f^{(n)}(b_i,\tau)\,, \tag{33}$$

evaluated at lattice points $b_i\in\Lambda_M$, that is, Eisenstein series for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ [6]:

$$\begin{aligned}
2\pi i\partial_\tau\,\omega\left(\begin{smallmatrix}n_1,\,\ldots,\,n_\ell\\ b_1,\,\ldots,\,b_\ell\end{smallmatrix}\right)&=h^{(n_\ell+1)}(-b_\ell)\,\omega\left(\begin{smallmatrix}n_1,\,\ldots,\,n_{\ell-1}\\ b_1,\,\ldots,\,b_{\ell-1}\end{smallmatrix}\right)-h^{(n_1+1)}(-b_1)\,\omega\left(\begin{smallmatrix}n_2,\,\ldots,\,n_\ell\\ b_2,\,\ldots,\,b_\ell\end{smallmatrix}\right)\\
&+\sum_{i=2}^{\ell}\Bigg[\theta_{n_i\geq 1}\sum_{k=0}^{n_{i-1}+1}\binom{n_i+k-1}{k}h^{(n_{i-1}-k+1)}(b_i-b_{i-1})\,\omega\left(\begin{smallmatrix}n_1,\,\ldots,\,n_{i-2},\,n_i+k,\,n_{i+1},\,\ldots,\,n_\ell\\ b_1,\,\ldots,\,b_{i-2},\,b_i,\,b_{i+1},\,\ldots,\,b_\ell\end{smallmatrix}\right)\\
&\quad-\theta_{n_{i-1}\geq 1}\sum_{k=0}^{n_i+1}\binom{n_{i-1}+k-1}{k}h^{(n_i-k+1)}(b_{i-1}-b_i)\,\omega\left(\begin{smallmatrix}n_1,\,\ldots,\,n_{i-2},\,n_{i-1}+k,\,n_{i+1},\,\ldots,\,n_\ell\\ b_1,\,\ldots,\,b_{i-2},\,b_{i-1},\,b_{i+1},\,\ldots,\,b_\ell\end{smallmatrix}\right)\\
&\quad+(-1)^{n_i+1}\theta_{n_{i-1}\geq 1}\theta_{n_i\geq 1}h^{(n_{i-1}+n_i+1)}(b_i-b_{i-1})\,\omega\left(\begin{smallmatrix}n_1,\,\ldots,\,n_{i-2},\,0,\,n_{i+1},\,\ldots,\,n_\ell\\ b_1,\,\ldots,\,b_{i-2},\,0,\,b_{i+1},\,\ldots,\,b_\ell\end{smallmatrix}\right)\Bigg]\,.
\end{aligned} \tag{34}$$

[4] Note that the normalization conventions of the functions $h^{(n)}(b,\tau)$ in eq. (33) and ref. [6] differ from the definition of the Eisenstein series $h^{(n)}_{M,r,s}=f^{(n)}(\frac{r}{M}+\frac{s}{M}\tau,\tau)$ for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in ref. [10].

We have introduced $\theta_{n\geq 1} = 1-\delta_{n,0}$ for non-negative n, indicating that $n_i = 0$ cause certain terms in the last three lines to vanish. For teMZVs of length $\ell > 1$ on the left-hand side of eq. (34), each teMZV on the right-hand side has lower length $\ell-1$. Hence, eq. (34) allows to recursively convert teMZVs to iterated integrals over the functions $h^{(k)}(b,\tau)$, terminating with a vanishing right-hand side for $\ell = 1$. Upon evaluation at fixed lattice points $b_i \in \Lambda_M$, the functions $h^{(k)}(b,\tau)$ are holomorphic in the modular parameter τ. For any $k > 2$, they can be conveniently represented as a lattice sum

$$h^{(k)}\Big(\frac{r}{M}+\tau\frac{s}{M},\tau\Big) = (k-1)\sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i\frac{rm-sn}{M}}}{(n+m\tau)^k}\,. \tag{35}$$

In order to render the corresponding expression finite for $k = 2$, the summation prescription has to be modified. Alternatively, level-M Eisenstein series have series expansions in $q^{1/M}$ [6], for example one finds

$$h^{(4)}\Big(\frac{\tau}{2},\tau\Big) = \frac{\zeta_4}{4}\Big(7-240q^{1/2}-240q-6720q^{3/2}-240q^2-30240q^{5/2}+\cdots\Big)\,. \tag{36}$$

For $r = s = 0$, one recovers the usual holomorphic Eisenstein series (cf. eq. (35))

$$\frac{h^{(k)}(0,\tau)}{1-k} = \mathrm{G}_k(\tau) = \sum_{(m,n)\neq(0,0)}\frac{1}{(n+m\tau)^k}\,, \qquad k\geq 3\,. \tag{37}$$

Correspondingly, eq. (34) reduces to the differential equation for eMZVs stated in eq. (2.47) of ref. [5]. Nicely, the situation $k = 2$ in the equation above does not occur, when considering the τ-derivative eq. (34) of convergent eMZVs.

Considering the differential equation (34) and the identification (37), one can finally rewrite every eMZV in terms of iterated integrals of Eisenstein series [5]:

$$\begin{aligned}\mathscr{E}_0(k_1,k_2,\ldots,k_r;q) &:= -\int_0^q \frac{\mathrm{d}q_r}{q_r}\frac{\mathrm{G}^0_{k_r}(q_r)}{(2\pi i)^{k_r}}\,\mathscr{E}_0(k_1,k_2,\ldots,k_{r-1};q_r) \\ &= (-1)^r \int\limits_{0\leq q_1\leq q_2\leq\ldots\leq q_r\leq q} \frac{\mathrm{d}q_1}{q_1}\cdots\frac{\mathrm{d}q_r}{q_r}\frac{\mathrm{G}^0_{k_1}(q_1)}{(2\pi i)^{k_1}}\cdots\frac{\mathrm{G}^0_{k_r}(q_r)}{(2\pi i)^{k_r}}\,.\end{aligned} \tag{38}$$

The recursion starts with $\mathscr{E}_0(;\tau) = 1$, and the non-constant parts of Eisenstein series are defined as

$$\mathrm{G}^0_{2n}(\tau) = \mathrm{G}_{2n}(\tau) - 2\,\zeta_{2n}\,, \qquad \mathrm{G}_0(\tau) = \mathrm{G}^0_0(\tau) = -1 \tag{39}$$

with $n\in\mathbb{N}$. For iterated integrals $\mathscr{E}_0(k_1,k_2,\ldots,k_r;q)$ in eq. (38), the number of non-zero entries ($k_j\neq 0$) is called the *depth* of the iterated Eisenstein integral.

The iterated Eisenstein integrals $\mathscr{E}_0(k_1,\ldots,k_r;q)$ with $k_1\neq 0$ are nicely convergent and do not need to be regularized. Even more, the conversion of untwisted eMZVs to iterated Eisenstein integrals provides an easy way to identify their relations [5, 30, 31]. Many of such eMZV relations are available in digital form [32] similar to the datamine of multiple zeta values [33].

In the same way as one can rewrite untwisted eMZVs as iterated integrals over the Eisenstein series eq. (37), one can rewrite teMZVs as iterated integrals over the level-M Eisenstein series defined in eq. (36). In contrast to the situation for usual holomorphic Eisenstein series, there are several linear relations between level-M Eisenstein series, which are discussed in ref. [10] and which need to taken into account when deriving functions relations in general. In the realm of string amplitudes discussed in the next subsection, we will however encounter only one particular Eisenstein series at level-2, which does not require these additional relations in order to reach a canonical representation.

3.2 Eisenstein series of level 2 in the string context

Although the differential equation (34) is applicable to Eisenstein series evaluated at points of any sublattice Λ_M, let us focus on the lattice Λ_2 suitable for string amplitudes. As will be elaborated in section 4, the parametrization of the cylinder worldsheet in figure 2 gives rise to teMZVs with twists $b \in \{0, \tau/2\}$ in the non-planar amplitudes. Hence, the differential equation (34) allows to express the α'-expansion in terms of iterated Eisenstein integrals involving $h^{(k)}\left(\frac{\tau}{2},\tau\right)$ and $h^{(k)}(0,\tau) = (1-k)\mathrm{G}_k(\tau)$.

When expressing the teMZVs from the non-planar integrals in terms of a basis of iterated Eisenstein integrals, the contributions from $h^{(k)}\left(\frac{\tau}{2},\tau\right)$ turn out to cancel. In other words, even for the non-planar integrals $I_{12|34}$ and $I_{123|4}$ of eq. (14), the α'-expansions shown in the next section are expressible in terms of untwisted eMZVs or iterated integrals over $\mathrm{G}_k(\tau)$ exclusively. In spite of the cancellation of all non-trivial twists, the representation of intermediate results in terms of Eisenstein series for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ has been indispensable to attain a canonical form for all contributions.

As an example for the τ-derivative in eq. (34), let us take the teMZVs

$$\begin{aligned} 2\pi i \frac{\partial}{\partial\tau}\,\omega\begin{pmatrix} 0, & 1, & 1 \\ 0, & \tau/2, & \tau/2 \end{pmatrix} &= h^{(2)}\left(\frac{\tau}{2},\tau\right)\omega\begin{pmatrix} 0, & 1 \\ 0, & \tau/2 \end{pmatrix} - \omega\begin{pmatrix} 2, & 1 \\ \tau/2, & \tau/2 \end{pmatrix} \\ 2\pi i \frac{\partial}{\partial\tau}\,\omega\begin{pmatrix} 0, & 1 \\ 0, & \tau/2 \end{pmatrix} &= h^{(2)}\left(\frac{\tau}{2},\tau\right) - \zeta_2\,. \end{aligned} \tag{40}$$

Since intermediate steps in the expansion of $I_{123|4}$ and $I_{12|34}$ turn out to involve the rigid combination $2\,\omega\begin{pmatrix} 0, & 1, & 1 \\ 0, & \tau/2, & \tau/2 \end{pmatrix} - \omega\begin{pmatrix} 0, & 1 \\ 0, & \tau/2 \end{pmatrix}^2$, the contribution of $h^{(2)}\left(\frac{\tau}{2},\tau\right)$ in eq. (40) cancels. Moreover, the relation $2\,\omega\begin{pmatrix} 0, & 1, & 1 \\ 0, & \tau/2, & \tau/2 \end{pmatrix} - \omega\begin{pmatrix} 0, & 1 \\ 0, & \tau/2 \end{pmatrix}^2 = \omega(0,0,2)+\frac{\zeta_2}{3}$ can be checked by taking higher τ-derivatives of the left-hand side.

Similarly, the τ-derivative eq. (34) and the decomposition described in the previous subsection yield

$$\begin{aligned}\omega(0,0,2) &= -\frac{\zeta_2}{3} - 6\,\mathcal{E}_0(4,0;\tau)\\ \omega(0,1,0,0) &= \frac{3\zeta_3}{4\pi^2} - \frac{9}{2\pi^2}\,\mathcal{E}_0(4,0,0;\tau)\\ \omega(0,3,0,0) &= 180\,\mathcal{E}_0(6,0,0;\tau)\,.\end{aligned} \tag{41}$$

The terms $-\frac{\zeta_2}{3}$ and $\frac{3\zeta_3}{4\pi^2}$ at the order of q^0 exemplify that integration constants have to be taken into account when expressing teMZVs as integrals over their τ-derivatives eq. (34). For the twists $b \in \{0, \tau/2\}$ of our interest, the integration constants are rational combinations of $(2\pi i)^{-1}$ and multiple zeta values that can be determined by the techniques in section 2.3 of [5] and section 3.2 of [6].

4 Four-point results in different languages

In this section, we apply the mathematical framework of section 3 to the α'-expansion of the four-point cylinder integrals eq. (14). In order to relate the Green function eq. (11) to the constituents of teMZVs, we use momentum conservation eq. (15) to rewrite the target integrals[5] as

$$\begin{aligned}I_{1234}(s_{ij},q) &= \int_0^1 \mathrm{d}x_4 \int_0^{x_4} \mathrm{d}x_3 \int_0^{x_3} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1\, \delta(x_1) \exp\Big(\sum_{i<j}^4 s_{ij} P(x_{ij},q)\Big)\\ I_{123|4}(s_{ij},q) &= \Big(\prod_{l=1}^4 \int_0^1 \mathrm{d}x_l\Big)\, \delta(x_1) \exp\Big(\sum_{i<j}^3 s_{ij} P(x_{ij},q) + \sum_{j=1}^3 s_{j4} Q(x_{ij},q)\Big)\\ I_{12|34}(s_{ij},q) &= q^{\frac{s_{12}}{4}} \Big(\prod_{l=1}^4 \int_0^1 \mathrm{d}x_l\Big)\, \delta(x_1) \exp\Big(\sum_{\substack{(i,j)=\\(1,2),(3,4)}} s_{ij} P(x_{ij},q) + \sum_{\substack{i=1,2\\j=3,4}} s_{ij} Q(x_{ij},q)\Big)\,,\end{aligned} \tag{42}$$

with the expressions

$$G(z,\tau), \quad \mathrm{Im}\, z = 0 \quad \rightsquigarrow \quad P(x,q) = \Gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix};x\right) - \omega(1,0) \tag{43}$$

$$G(z,\tau), \quad \mathrm{Im}\, z = \frac{\mathrm{Im}\,\tau}{2} \quad \rightsquigarrow \quad Q(x,q) = \Gamma\left(\begin{smallmatrix}1\\ \tau/2\end{smallmatrix};x\right) - \omega\left(\begin{smallmatrix}1, & 0\\ \tau/2, & 0\end{smallmatrix}\right)\,, \tag{44}$$

where $x = \mathrm{Re}\, z$, and the Green functions $P(x,q)$ and $Q(x,q)$ connect punctures on the same and on different cylinder boundaries, respectively. Both summands $\Gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix};x\right)$ and $\omega(1,0)$ in eq. (43) individually represent divergent integrals whose regularization is discussed in detail in section 4.2 of [6]. As visualized in figure 2, the twists $\tau/2$ in eq. (44) stem from the displacement of the two cylinder boundaries in our

[5] The derivation of eq. (42) from eq. (14) is elaborated on in ref. [6]. The only difference is that the present definitions of $P(x,q)$ and $Q(x,q)$ in eqs. (43) and (44) deviate from those in the reference by an additive constant. Instead, the objects $P(x,q)$ and $Q(x,q)$ defined in eqs. (43) and (44) match the expressions in ref. [11] up to an overall minus sign.

parametrization through a rectangular torus. Accordingly, the factor of $q^{s_{12}/4}$ in the above expression for the non-planar contribution $I_{12|34}(s_{ij},q)$ can be traced back to the term $\sim (\mathrm{Im}\, z)^2$ in the Green function eq. (11).

When inserting the differences $x_{ij} = x_i - x_j$ of the cylinder punctures into the Green functions $P(x,q)$ and $Q(x,q)$, the following representations turn out to be particularly convenient for the α'-expansion of eq. (42)

$$P(x_{ij},q) = \Gamma\left(\begin{smallmatrix}1\\ x_j\end{smallmatrix};x_i\right) + \Gamma\left(\begin{smallmatrix}1\\ 0\end{smallmatrix};x_j\right) - \omega(1,0)\,, \qquad 1<i<j \tag{45}$$

$$Q(x_{ij},q) = \Gamma\left(\begin{smallmatrix}1\\ x_j+\tau/2\end{smallmatrix};x_i\right) + \Gamma\left(\begin{smallmatrix}1\\ \tau/2\end{smallmatrix};x_j\right) - \omega\left(\begin{smallmatrix}1, & 0\\ \tau/2, & 0\end{smallmatrix}\right)\,, \qquad 1<i<j\,. \tag{46}$$

4.1 The proof of concept

The α'-expansion of the open-string integrals eq. (42) at fixed[6] τ can be obtained by Taylor-expanding the exponentials in the integrand w.r.t. s_{ij} and employing the representations of the Green functions in eq. (43) to eq. (46). The order-by-order integration can be reduced to the definitions of elliptic iterated integrals and teMZVs in section 3 as soon as the following technical subtleties have been settled:

- The recursive definition eq. (30) of elliptic iterated integrals cannot be used for integrands of the form $\mathrm{d}t\, f^{(n)}(t-b_1) f^{(m)}(t-b_2)$ with multiple occurrence of the integration variable t as arguments of different integration kernels in eq. (26). This situation can be remedied by using the Fay relation (29), which can be viewed as the elliptic analogue of partial-fraction relations $\frac{1}{(t-b_1)(t-b_2)} + \mathrm{cyc}(t,b_1,b_2) = 0$. Then, each term on the right-hand side of the Fay relation can be recursively integrated via eq. (30).
- The integration variable of eq. (30) is not allowed to show up in the shifts b_i of the iterated integral Γ in the integrand. Therefore one has to derive functional relations between different iterated integrals. The main mechanism to derive relations like

$$\Gamma\left(\begin{smallmatrix}3,1\\ 0,z\end{smallmatrix};z\right) = -4\Gamma\left(\begin{smallmatrix}0,4\\ 0,0\end{smallmatrix};z\right) + \Gamma\left(\begin{smallmatrix}1,3\\ 0,0\end{smallmatrix};z\right) - \Gamma\left(\begin{smallmatrix}2,2\\ 0,0\end{smallmatrix};z\right) - \Gamma\left(\begin{smallmatrix}4,0\\ 0,0\end{smallmatrix};z\right) \tag{47}$$

  consists of writing Γ as an integral over its own z-derivative and using again Fay relations on the integration kernels $f^{(n)}$ before integrating back [4]. The need for relations like eq. (47) arises less frequently if the representations eqs. (43) and (44) are used for propagators at argument x_{1j} with $j \neq 1$ and eqs. (45) and (46) for propagators at argument x_{ij} with $1<i<j$.

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[6] Given that the α'-expansions in this work are performed at fixed τ, our results do not expose the branch cuts of the loop amplitudes which result from the integral over q in eqs. (13) and (19). In the terminology of the closed-string literature [12], the analysis of these proceedings is restricted to the analytic dependence of the one-loop amplitudes on the kinematic invariants.

- The association of $1 < i < j$ with eqs. (45) and (46) is adapted to an integration region where $0 < x_2 < x_3 < x_4 < 1$. The non-planar integrals $I_{123|4}$ and $I_{12|34}$, however, additionally involve situations where $x_j > x_{j+1}$. Still, the cubical integration region $x_{j=2,3,4} \in (0,1)$ of $I_{123|4}$ and $I_{12|34}$ can be decomposed into six simplices $0 < x_i < x_j < x_k < 1$ with some permutations (i,j,k) of $(2,3,4)$. Each of these simplicial contributions in turn can then be reduced to the situation where $0 < x_2 < x_3 < x_4 < 1$ by simultaneous relabeling of the integration variables and the Mandelstam variables s_{ij}.

Further details and examples of this rather technical procedure can be found in refs. [4, 6, 11]. For the purpose of these proceedings, let us just note that all integrals resulting from the α'-expansion of the integrand in eq. (42) can be treated in this way; thus integration using eq. (30) is possible.

Since the upper limit for the outermost integration in each term of eq. (42) is $x_j = 1$, the elliptic iterated integrals in the α'-expansions ultimately boil down to teMZVs eq. (31). Once the punctures x_2, x_3, x_4 are all integrated out, the leftover shifts b_j can take the values 0 and $\tau/2$. In the planar case I_{1234} with all integrations on the same boundary, there are no shifts; thus the α'-expansions are manifestly expressible in terms of untwisted eMZVs eq. (32).

Note that the representation of the Green function used in the first discussion of the planar case [4] did not involve the subtraction of $\omega(1,0)$ in eqs. (43) and (45). As a virtue of the Green function $P(x_{ij},q)$ including $-\omega(1,0)$, divergent eMZVs $\omega(1,\ldots)$ or $\omega(\ldots,1)$ (cf. the discussion prior to section 3.1) automatically cancel from the α'-expansion along with each monomial in the s_{ij}. In other words, short-distance finiteness of the integrals is manifest term by term[7] without further use of momentum conservation.

Finally, the expansion of the non-planar integrals benefits from the particular choice of Green functions in eqs. (43) and (44): The vanishing of $\int_0^1 \mathrm{d}x\, P(x,q)$ and $\int_0^1 \mathrm{d}x\, Q(x,q)$ [11] systematically bypasses various spurious terms, which appear in intermediate steps when using the representation of Green functions from ref. [6].

4.2 Plain results

Following the steps outlined in the previous section, the α'-expansion of the integral I_{1234} for the planar four-point cylinder amplitude eq. (13) can be brought into the following form [4]

$$\begin{aligned} I_{1234}(s_{ij},q) \;=& \frac{1}{6} + 2\,\omega(0,1,0,0)\,s_{13} \\ & + 2\,\omega(0,1,1,0,0)\big(s_{12}^2 + s_{23}^2\big) - 2\,\omega(0,1,0,1,0)\,s_{12}s_{23} \end{aligned} \tag{48}$$

[7] For instance, the contributions $s_{12}P(x_{12},q)$ and $s_{13}P(x_{13},q)$ from the exponentials in the representation eq. (42) of $I_{1234}(s_{ij},q)$ integrate to $\omega(1,0,0,0) - \frac{1}{6}\,\omega(1,0) = -\frac{1}{3}\,\omega(0,1,0,0)$ and $\omega(1,0,0,0) + \omega(0,1,0,0) - \frac{1}{6}\,\omega(1,0) = \frac{2}{3}\,\omega(0,1,0,0)$, respectively.

$$+\,\beta_5\,(s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3)\,-\,\beta_{2,3}\,s_{12}s_{23}s_{13}+\mathscr{O}(\alpha'^4)\,,$$

where we have used the following shorthands for the third order in α'

$$\begin{aligned}\beta_5 \;&=\; \frac{4}{3}\big[\omega(0,0,1,0,0,2)+\omega(0,1,1,0,1,0)-\omega(2,0,1,0,0,0)-\zeta_2\,\omega(0,1,0,0)\big]\\ \beta_{2,3} \;&=\; \frac{\zeta_3}{12}+\frac{8\zeta_2}{3}\,\omega(0,1,0,0)-\frac{5}{18}\,\omega(0,3,0,0)\,. \end{aligned} \tag{49}$$

In the non-planar four-point integrals of eq. (13), the teMZVs obtained in intermediate steps are found to cancel by employing the canonical representation in terms of iterated Eisenstein integrals. With two punctures on each boundary, the cancellations of teMZVs in

$$\begin{aligned}q^{-\frac{s_{12}}{4}}I_{12|34}(s_{ij},q)&=1+s_{12}^2\Big(\frac{7\zeta_2}{6}+2\,\omega(0,0,2)\Big)-2\,s_{13}s_{23}\Big(\frac{\zeta_2}{3}+\omega(0,0,2)\Big)\\ &-4\zeta_2\,\omega(0,1,0,0)s_{12}^3+s_{12}s_{13}s_{23}\left(\frac{5}{3}\,\omega(0,3,0,0)+4\zeta_2\,\omega(0,1,0,0)-\frac{\zeta_3}{2}\right)+\mathscr{O}(\alpha'^4)\end{aligned} \tag{50}$$

are guaranteed to extend to all orders in α' by the factorization argument in section 4.3.5 of [6]. The other non-planar topology with three punctures on the same boundary exhibits the same kinds of cancellations [6]

$$\begin{aligned}I_{123|4}(s_{ij},q)&=1+(s_{12}^2+s_{12}s_{23}+s_{23}^2)\left(\frac{7\zeta_2}{6}+2\omega(0,0,2)\right)\\ &-s_{12}s_{23}s_{13}\left(4\zeta_2\,\omega(0,1,0,0)-\frac{5}{3}\,\omega(0,3,0,0)+\frac{\zeta_3}{2}\right)+\mathscr{O}(\alpha'^4)\end{aligned} \tag{51}$$

which might have an all-order explanation from the monodromy relations [34–36] among one-loop open-string amplitudes. The above results have been checked to reproduce the degeneration limits $q\to 0$ known from the literature, i.e. the zero'th order in the q-expansions of $I_{1234}(s_{ij},q), I_{123|4}(s_{ij},q)$ and $q^{-\frac{s_{12}}{4}}I_{12|34}(s_{ij},q)$ agrees with the expressions in refs. [37] and [35], respectively.

4.3 Results in terms of iterated Eisenstein integrals

In this section, we rewrite the above α'-expansions in a canonical form by converting the eMZVs to a basis of iterated Eisenstein integrals (38). The planar integral eq. (48) then takes the form

$$\begin{aligned}I_{1234}(s_{ij},q)&=\frac{1}{6}+\frac{3s_{13}}{2\pi^2}\big(\zeta_3-6\mathscr{E}_0(4,0,0;q)\big)+(s_{12}^2+s_{12}s_{23}+s_{23}^2)\Big(\frac{\zeta_2}{6}-2\mathscr{E}_0(4,0;q)\Big)\\ &+\frac{1}{\pi^2}(s_{12}^2+4s_{12}s_{23}+s_{23}^2)\Big(60\mathscr{E}_0(6,0,0,0;q)-\frac{\zeta_4}{2}\Big)\end{aligned}$$

$$
\begin{aligned}
&+s_{12}s_{13}s_{23}\Big(2\mathcal{E}_0(4,0,0;q)+50\mathcal{E}_0(6,0,0;q)-\frac{5\zeta_3}{12}\Big) \qquad (52)\\
&+\frac{1}{\pi^2}(s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3)\Big(216\mathcal{E}_0(4,0,4,0,0;q)+648\mathcal{E}_0(4,4,0,0,0;q)\\
&\quad+\frac{3}{5}\mathcal{E}_0(4,0,0,0,0;q)-108\mathcal{E}_0(4,0;q)\mathcal{E}_0(4,0,0;q)+2016\mathcal{E}_0(8,0,0,0,0;q)\\
&\quad+18\mathcal{E}_0(4,0;q)\zeta_3-\frac{5\zeta_5}{2}\Big)+\mathcal{O}(\alpha'^4)\,,
\end{aligned}
$$

where the third order in α' exhibits integrals $\mathcal{E}_0(4,4,0,0,0;q)$ and $\mathcal{E}_0(4,0,4,0,0;q)$ of depth two. The non-planar integral eq. (50) in turn contains shorter eMZVs and iterated Eisenstein integrals at the orders under consideration, cf. eq. (41),

$$
\begin{aligned}
q^{-\frac{s_{12}}{4}}I_{12|34}(s_{ij},q)&=1+s_{12}^2\Big(\frac{\zeta_2}{2}-12\mathcal{E}_0(4,0;q)\Big)+12\,s_{13}s_{23}\mathcal{E}_0(4,0;q) \qquad (53)\\
&+s_{12}^3\Big(3\mathcal{E}_0(4,0,0;q)-\frac{\zeta_3}{2}\Big)+s_{12}s_{13}s_{23}\Big(300\mathcal{E}_0(6,0,0;q)-3\mathcal{E}_0(4,0,0;q)\Big)+\mathcal{O}(\alpha'^4)\,,
\end{aligned}
$$

and a similar structure can be found for eq. (51):

$$
\begin{aligned}
I_{123|4}(s_{ij},q)&=1+(s_{12}^2+s_{12}s_{23}+s_{23}^2)\Big(\frac{\zeta_2}{2}-12\mathcal{E}_0(4,0;q)\Big) \qquad (54)\\
&+s_{12}s_{23}s_{13}\Big(300\mathcal{E}_0(6,0,0;q)+3\mathcal{E}_0(4,0,0;q)-\zeta_3\Big)+\mathcal{O}(\alpha'^4)\,.
\end{aligned}
$$

Note that the α'-expansions of both non-planar integrals $q^{-\frac{s_{12}}{4}}I_{12|34}(s_{ij},q)$ and $I_{123|4}(s_{ij},q)$ take a form very similar to the symmetrized version of the planar integral eq. (52):

$$
\begin{aligned}
I_{1234}(s_{ij},q)+\mathrm{perm}(2,3,4)&=1+(s_{12}^2+s_{12}s_{23}+s_{23}^2)\Big(\zeta_2-12\mathcal{E}_0(4,0;q)\Big) \qquad (55)\\
&+s_{12}s_{23}s_{13}\Big(12\mathcal{E}_0(4,0,0;q)+300\mathcal{E}_0(6,0,0;q)-\frac{5\zeta_3}{2}\Big)+\mathcal{O}(\alpha'^4)\,.
\end{aligned}
$$

In fact, taking the differences between eq. (55) and eq. (53) or (54), they are proportional to ζ_2, which might become visible only after using relations like $\zeta_2\,\omega(0,1,0,0)=\frac{\zeta_3}{8}-\frac{3}{4}\mathcal{E}_0(4,0,0)$. This observation is related to the expectation on the corresponding closed-string integral [12, 13] to follow from open-string quantities under a suitably chosen single-valued projection: The agreement of eq. (55) and eq. (53) or (54) modulo ζ_2 is argued in ref. [11] to pave the way towards a tentative single-valued projection for eMZVs.

While there is no bottleneck in obtaining higher orders in α' from the same methods, it would be desirable to construct cylinder integrals directly from the elliptic associators [38]. This would generalize the representations of disk integrals in terms of the Drinfeld associator [39] and should explain the patterns of iterated Eisenstein integrals in the above equations.

5 Five-point results in different languages

In this section, we discuss the applicability of the setup of teMZVs to string amplitudes of multiplicities higher than four. The main novelties for maximally supersymmetric amplitudes at $n \geq 5$ points are kinematic poles of the worldsheet integrals and higher-dimensional bases of tensor structures for the external polarizations. The appearance of both of these features are captured by the subsequent discussion of five-point one-loop amplitudes of the open superstring.

We will focus on the α'-expansion of the prototype integrals in eq. (20) to (23) which are more conveniently written in terms of the propagators in eq. (43) to (46),

$$H^{12}_{12345}(s_{ij},q) = \int_0^1 \mathrm{d}x_5 \Big(\prod_{l=1}^{4} \int_0^{x_{l+1}} \mathrm{d}x_l \Big) \, \delta(x_1) \, f^{(1)}_{12} \, \exp\Big(\sum_{i<j}^{5} s_{ij} P(x_{ij}) \Big) \tag{56}$$

$$\widehat{H}^{13}_{12345}(s_{ij},q) = \int_0^1 \mathrm{d}x_5 \Big(\prod_{l=1}^{4} \int_0^{x_{l+1}} \mathrm{d}x_l \Big) \, \delta(x_1) \, f^{(1)}_{13} \, \exp\Big(\sum_{i<j}^{5} s_{ij} P(x_{ij}) \Big) \tag{57}$$

$$\begin{aligned} H^{12}_{123|45}(s_{ij},q) = q^{\frac{s_{45}}{4}} \Big(\prod_{l=3}^{5} \int_0^1 \mathrm{d}x_l \Big) \int_0^{x_3} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1 \, \delta(x_1) \, f^{(1)}_{12} \\ \times \exp\Big(\sum_{i<j}^{3} s_{ij} P(x_{ij}) + s_{45} P(x_{45}) + \sum_{\substack{i=1,2,3 \\ j=4,5}} s_{ij} Q(x_{ij}) \Big) \end{aligned} \tag{58}$$

$$\begin{aligned} \widehat{H}^{14}_{123|45}(s_{ij},q) = q^{\frac{s_{45}}{4}} \Big(\prod_{l=3}^{5} \int_0^1 \mathrm{d}x_l \Big) \int_0^{x_3} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1 \, \delta(x_1) \, f^{(1)}_{14} \\ \times \exp\Big(\sum_{i<j}^{3} s_{ij} P(x_{ij}) + s_{45} P(x_{45}) + \sum_{\substack{i=1,2,3 \\ j=4,5}} s_{ij} Q(x_{ij}) \Big) \, . \end{aligned} \tag{59}$$

5.1 Kinematic poles

When reproducing field-theory amplitudes from the $\alpha' \to 0$ limit of string theories, Feynman propagators arise from the boundaries of the moduli spaces. For instance, the s-channel pole in a four-point open-string tree amplitude arises from the region in the disk integral

$$\int_0^1 \frac{\mathrm{d}z_2}{z_2} \, z_2^{s_{12}} \, (1-z_2)^{s_{23}} = \frac{1}{s_{12}} + \mathscr{O}(\alpha') \, , \tag{60}$$

where the puncture z_2 collides with $z_1 = 0$. Since the emergence of kinematic poles s_{ij}^{-1} is solely dictated by local properties of the worldsheet and the short-distance behavior of the Green function, the pole structure of loop amplitudes can be analyzed by the same methods as their tree-level counterparts[8].

In contrast to the one-loop four-point integrands, the prototype integrals at five points in eq. (56) to (59) exhibit additional factors of $f^{(1)}_{ij}$ with

$$f^{(1)}_{ij} = \frac{1}{z_i - z_j} + \mathscr{O}(|z_i - z_j|) \tag{61}$$

which modify the singularity structure at the boundary of the moduli space. In particular, the worldsheet singularities of $f^{(1)}_{12} \mathrm{e}^{s_{12} P(x_{12})}$ translate into kinematic poles $\sim s_{12}^{-1}$ in the five-point one-loop integrals eqs. (56) and (58) along the lines of the tree-level mechanism in eq. (60). As a convenient way of capturing the α'-expansion of such singular integrals, we split the integrand of H^{12}_{12345} in eq. (56) as

$$\begin{aligned} f^{(1)}_{12}\, \mathrm{e}^{\sum_{i<j}^5 s_{ij} P(x_{ij})} &= f^{(1)}_{12}\, \mathrm{e}^{s_{12} P(x_2)} \big[\Phi(x_2,x_3,x_4,x_5) - \Phi(0,x_3,x_4,x_5) + \Phi(0,x_3,x_4,x_5) \big] \\ \Phi(x_2,x_3,x_4,x_5) &= \exp\Big(\sum_{l=3}^{5} s_{1l} P(x_l) + \sum_{2 \leq i<j}^{5} s_{ij} P(x_{ij}) \Big) , \end{aligned} \tag{62}$$

where we remind the reader that we fixed $x_1 = 0$. Then, for the last term of the first line, the integral over x_2 becomes elementary by recognizing $f^{(1)}_{12}\, \mathrm{e}^{s_{12} P(x_2)} = -\frac{1}{s_{12}} \frac{\partial}{\partial x_2} \mathrm{e}^{s_{12} P(x_2)}$ and leads to the following singular part of H^{12}_{12345}:

$$\begin{aligned} &\int_0^1 \mathrm{d}x_5 \Big(\prod_{l=1}^{4} \int_0^{x_{l+1}} \mathrm{d}x_l \Big)\, \delta(x_1)\, f^{(1)}_{12}\, \mathrm{e}^{s_{12} P(x_2)}\, \Phi(0,x_3,x_4,x_5) \\ &= -\frac{1}{s_{12}} \int_0^1 \mathrm{d}x_5 \int_0^{x_5} \mathrm{d}x_4 \int_0^{x_4} \mathrm{d}x_3 \exp\Big(s_{12} P(x_3) + \sum_{l=3}^{5} (s_{1l} + s_{2l}) P(x_l) + \sum_{3 \leq i<j}^{5} s_{ij} P(x_{ij}) \Big) . \end{aligned} \tag{63}$$

The right-hand side of the equation above can in turn be identified with the planar four-point integral in eq. (14) after relabeling the Mandelstam invariants as

$$\chi : \begin{cases} s_{12} \to s_{123} , & s_{13} \to s_{14} + s_{24} , \quad s_{14} \to s_{15} + s_{25} \\ s_{23} \to s_{34} , & s_{24} \to s_{35} , \quad\quad\;\; s_{34} \to s_{45} \end{cases} \tag{64}$$

with $s_{123} = s_{12} + s_{13} + s_{23}$. We have assumed s_{12} to have a positive real part in discarding the boundary term $\mathrm{e}^{s_{12} P(x_2)} \big|_{x_2=0}$ in eq. (63) which exhibits the same short-distance behavior $x_2^{s_{12}}$ as seen in the tree-level integrand eq. (60). Hence, the integral eq. (20) can be split into a pole part and a regular part according to

$$H^{12}_{12345} = H^{12,\mathrm{reg}}_{12345} - \frac{I_{1234}(\chi(s_{ij}), q)}{s_{12}} \tag{65}$$

[8] See [40, 41] for two related approaches to treat the poles of n-point open-string tree amplitudes.

$$H_{12345}^{12,\text{reg}} = \int_0^1 \mathrm{d}x_5 \Big(\prod_{l=2}^4 \int_0^{x_{l+1}} \mathrm{d}x_l \Big) f^{(1)}(x_2)\, \mathrm{e}^{s_{12}P(x_2)} \big[\Phi(0,x_3,x_4,x_5) - \Phi(x_2,x_3,x_4,x_5) \big] \, .$$

In reconstructing the α'-expansion of the polar part from a four-point computation, the Mandelstam invariants of I_{1234} have to be transformed according to eq. (64) instead of using four-point momentum conservation eq. (15). This is the reason for obtaining

$$\begin{aligned} I_{1234}(\chi(s_{ij}),q) &= \frac{1}{6} + \omega(0,1,0,0)(s_{12}-2s_{34}-2s_{45}) \\ &+ \omega(0,1,1,0,0)(s_{12}^2-2s_{12}s_{34}+2s_{34}^2+2s_{45}^2) + \omega(0,1,0,1,0)(s_{12}-2s_{34})s_{45} + \mathscr{O}(\alpha'^3) \end{aligned} \tag{66}$$

instead of eq. (48) after using five-point momentum conservation eq. (25). The non-planar integral $H_{123|45}^{12}$ with a kinematic pole defined in eq. (58) will be addressed by a similar decomposition of the integrand as in eq. (62)

$$\begin{aligned} f_{12}^{(1)}\, \mathrm{e}^{\sum_{i<j}^5 s_{ij}P(x_{ij})} &= f_{12}^{(1)}\, \mathrm{e}^{s_{12}P(x_2)} \big[\Psi(x_2,x_3,x_4,x_5) - \Psi(0,x_3,x_4,x_5) + \Psi(0,x_3,x_4,x_5) \big] \\ \Psi(x_2,x_3,x_4,x_5) &= \exp\Big(s_{13}P(x_3) + \sum_{\substack{(i,j)= \\ (2,3),(4,5)}} s_{ij}P(x_{ij}) + \sum_{j=4,5} s_{1j}Q(x_j) + \sum_{\substack{i=2,3 \\ j=4,5}} s_{ij}Q(x_{ij}) \Big) \, . \end{aligned} \tag{67}$$

Again, one can find a primitive w.r.t. x_2 for the last term in the first line and arrive at a decomposition analogous to eq. (65)

$$\begin{aligned} H_{123|45}^{12} &= H_{123|45}^{12,\text{reg}} - \frac{I_{12|34}(\chi(s_{ij}),q)}{s_{12}} \\ H_{123|45}^{12,\text{reg}} &= q^{\frac{s_{45}}{4}} \Big(\prod_{l=3}^5 \int_0^1 \mathrm{d}x_l \Big) \int_0^{x_3} \mathrm{d}x_2\, f^{(1)}(x_2)\, \mathrm{e}^{s_{12}P(x_2)} \big[\Psi(0,x_3,x_4,x_5) - \Psi(x_2,x_3,x_4,x_5) \big] \, , \end{aligned} \tag{68}$$

with the same mapping eq. (64) of the Mandelstam invariants that governed the planar counterpart H_{12345}^{12}. The function $I_{12|34}(\chi(s_{ij}),q)$ of five-particle Mandelstam invariants along with s_{12}^{-1} is still expressible in terms of untwisted eMZVs,

$$\begin{aligned} I_{12|34}(\chi(s_{ij}),q) &= q^{\frac{s_{45}}{4}} \Big\{ 1 + s_{45}^2 \Big(\omega(0,0,2) + \frac{5\zeta_2}{6} \Big) \\ &+ \frac{1}{2} \big[(s_{14}+s_{24})^2 + (s_{15}+s_{25})^2 + s_{34}^2 + s_{35}^2 \big] \Big(\omega(0,0,2) + \frac{\zeta_2}{3} \Big) + \mathscr{O}(\alpha'^3) \Big\} \, , \end{aligned} \tag{69}$$

see eq. (53) for the analogous four-point expansion.

5.2 The regular parts

For the regular parts $H^{12,\text{reg}}_{12345}$ and $H^{12,\text{reg}}_{123|45}$ of the five-point integrals over $f^{(1)}_{12}$ defined in eqs. (65) and (68), the integrands

$$\begin{aligned}\Phi(0,x_3,x_4,x_5)-\Phi(x_2,x_3,x_4,x_5) &= -\sum_{j=3}^{5} s_{2j}\,\Gamma\left(\begin{smallmatrix}1\\ x_j\end{smallmatrix};x_2\right)+\mathscr{O}(\alpha'^2)\\ \Psi(0,x_3,x_4,x_5)-\Psi(x_2,x_3,x_4,x_5) &= -s_{23}\,\Gamma\left(\begin{smallmatrix}1\\ x_3\end{smallmatrix};x_2\right)-s_{24}\,\Gamma\left(\begin{smallmatrix}1\\ x_4+\tau/2\end{smallmatrix};x_2\right)\\ &\quad -s_{25}\,\Gamma\left(\begin{smallmatrix}1\\ x_5+\tau/2\end{smallmatrix};x_2\right)+\mathscr{O}(\alpha'^2)\end{aligned} \tag{70}$$

manifestly vanish as $x_2 \to 0$. Hence, they cancel the singularity of the integrands $f^{(1)}(x_2)$ in eqs. (65) and (68), and the integrations over x_3,x_4,x_5 yield convergent eMZVs at all orders, starting with[9]

$$H^{12,\text{reg}}_{12345} = (s_{23}-s_{25})\big[\,\omega(0,1,0,1,0)+2\,\omega(0,1,1,0,0)\big]+\mathscr{O}(\alpha'^2) \tag{71}$$

$$H^{12,\text{reg}}_{123|45} = \mathscr{O}(\alpha'^2)\,. \tag{72}$$

The leading three orders in the low-energy expansion of the planar integral H^{12}_{12345} can then be assembled by inserting eqs. (66) and (71) into eq. (65). Likewise, the non-planar integral $H^{12}_{123|45}$ follows from plugging eqs. (69) and (72) into eq. (68).

The kinematic poles of the integrals H^{12}_{12345} and $H^{12}_{123|45}$ only arise because the variables x_1 and x_2 of the worldsheet singularity $f^{(1)}_{12} \sim x^{-1}_{12}$ are neighbors in the integration domain $0<x_2<x_3<x_4<x_5<1$. In contrast, the integrals $\widehat{H}^{13}_{12345}$ and $\widehat{H}^{14}_{123|45}$ in eqs. (57) and (59), do not acquire any kinematic pole in this way. Accordingly, Taylor expanding the exponentials in the integrand automatically yields convergent eMZVs order by order in α' upon integration over x_2,x_3,x_4,x_5, e.g.

$$\begin{aligned}\widehat{H}^{13}_{12345} &= -\,\omega(0,1,0,0)+(s_{12}+s_{23}+s_{45})\,\omega(0,1,1,0,0)\\ &\quad +(s_{12}-s_{15}+s_{23}-s_{34}-s_{45})\,\omega(0,1,0,1,0)+\mathscr{O}(\alpha'^2)\end{aligned} \tag{73}$$

$$\widehat{H}^{14}_{123|45} = q^{\frac{s_{45}}{4}}\left\{(s_{24}-s_{34})\left(\omega(0,0,2)+\frac{\zeta_2}{3}\right)+\mathscr{O}(\alpha'^2)\right\}. \tag{74}$$

Note that to the orders considered, the α'-expansions of the five-point integrals can be easily confirmed to preserve the integration-by-parts relations

---

[9] The convergent integrals leading to eq. (72) can be performed via rearrangements such as [4] $\Gamma\left(\begin{smallmatrix}1&1\\0&z\end{smallmatrix};z\right)=-2\,\Gamma\left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix};z\right)-\Gamma\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix};z\right)-\zeta_2$, which is yet another example from the class of identities discussed around eq. (47). Note that the singular integration kernels $f^{(1)}$ manifestly drop out from this identity.

$$
\begin{aligned}
0 &= \int\limits_{D(\lambda)} \prod_{j=1}^{5} \mathrm{d}z_j \, \frac{\partial}{\partial z_2} \, \delta(z_1) \prod_{i<j}^{5} \exp\big(\tfrac{s_{ij}}{2} G(z_{ij}) \big) \qquad (75) \\
&= \int\limits_{D(\lambda)} \prod_{j=1}^{5} \mathrm{d}z_j \, \delta(z_1) \big[s_{23} f_{23}^{(1)} + s_{24} f_{24}^{(1)} + s_{25} f_{25}^{(1)} - s_{12} f_{12}^{(1)} \big] \prod_{i<j}^{5} \exp\big(\tfrac{s_{ij}}{2} G(z_{ij}) \big) \, .
\end{aligned}
$$

Such relations are crucial for manifesting the gauge invariance of the string amplitude. They do not depend on the planar or non-planar ordering λ in the integration region $D(\lambda)$, cf. eq. (6). Each of the summands in eq. (75) is expressible as a relabeling of one of the prototype integrals in eq. (56) to (59). It does not require much effort to show that non-planar integrals with a domain of the form $D(1,2,3,4|5)$ can be expanded using the same methods.

5.3 Putting everything together

Given the low-energy expansion of all the permutation-inequivalent prototype integrals eq. (56) to (59), one can expand the five-point cylinder amplitude eq. (19) at the level of the integrand w.r.t. q: The coefficients $I_{\lambda}^{\rho(2,3)}(s_{ij},q)$ of the independent kinematic factors $A^{\text{tree}}_{\text{SYM}}(1,\rho(2,3),4,5)$ with permutation $\rho \in S_2$ are linear combinations of the H_{λ}^{ij} and $\widehat{H}_{\lambda}^{ij}$ implicitly defined by combining eqs. (16) and (18) with BCJ relations of the $A^{\text{tree}}_{\text{SYM}}$.

Also the five-point tree amplitudes of the open superstring can be expanded in a BCJ basis of (super-)Yang–Mills amplitudes [42]: When considering the two single-trace orderings $A^{\text{tree}}_{\text{open}}(1,\tau(2,3),4,5)$ of disk amplitudes, the relation to their field-theory counterparts $A^{\text{tree}}_{\text{SYM}}(1,\rho(2,3),4,5)$ is encoded in 2×2 matrices $(P_w)_{\tau}{}^{\rho}$ and $(M_w)_{\tau}{}^{\rho}$ indexed by the permutations $\tau,\rho \in S_2$ [43],

$$
\begin{aligned}
A^{\text{tree}}_{\text{open}}(1,\tau(2,3),4,5) = \sum_{\rho \in S_2} &\big(1 + \zeta_2 P_2 + \zeta_3 M_3 + \zeta_2^2 P_4 + \mathscr{O}(\alpha'^5)\big)_{\tau}{}^{\rho} \\
&\times A^{\text{tree}}_{\text{SYM}}(1,\rho(2,3),4,5) \, . \qquad (76)
\end{aligned}
$$

The entries of the 2×2 matrices P_w and M_w are degree-w polynomials in s_{ij} with rational coefficients, e.g.

$$
P_2 = \begin{pmatrix} s_{12}s_{34} - s_{34}s_{45} - s_{51}s_{12} & s_{13}s_{24} \\ s_{12}s_{34} & s_{13}s_{24} - s_{24}s_{45} - s_{51}s_{13} \end{pmatrix} , \qquad (77)
$$

and analogous expressions for matrices at higher order in α' or multiplicity can be downloaded from [44].

The same matrices P_2, M_3, P_4 governing the low-energy expansion of tree amplitudes eq. (76) can be found in the planar sector at one loop: It is convenient to focus on the two choices $\lambda = 1,2,3,4,5$ and $\lambda = 1,3,2,4,5$ of the single-trace ordering which line up with the basis $A^{\text{tree}}_{\text{SYM}}(1,\rho(2,3),4,5)$ of kinematic factors in eq. (19). Doing so, the α'-expansions of the planar integrals H^{12}_{12345} and $\widehat{H}^{13}_{12345}$ uplift the relation eq. (76) between open-string and (super-)Yang–Mills tree-level amplitudes to one loop

$$A_{\text{cyl}}(1,\tau(2,3),4,5) = \int_0^1 \frac{\mathrm{d}q}{q} \sum_{\rho\in S_2} I_{1\tau(23)45}{}^{\rho}(s_{ij},q) A^{\text{tree}}_{\text{SYM}}(1,\rho(2,3),4,5) \tag{78}$$

with the leading low-energy orders [4]

$$\begin{aligned} -I_{1\tau(23)45}{}^{\rho}(s_{ij},q) = \frac{1}{6}P_2 &+ \Big(\frac{3\zeta_3}{2\pi^2} - \frac{9\mathscr{E}_0(4,0,0;q)}{\pi^2}\Big)M_3 \\ &+ \Big(\frac{\pi^2}{18} - 5\mathscr{E}_0(4,0;q) + \frac{150}{\pi^2}\mathscr{E}_0(6,0,0,0;q)\Big)P_4 \\ &+ \Big(\frac{3}{2}\mathscr{E}_0(4,0;q) - \frac{225}{\pi^2}\mathscr{E}_0(6,0,0,0;q)\Big)L_4 + \mathscr{O}(\alpha'^5)\,. \end{aligned} \tag{79}$$

At order α'^4, we encounter a new matrix L_4 with entries

$$\begin{aligned} (L_4)_{23}{}^{23} = s_{12}^2s_{23}^2 &+ 2s_{12}^2s_{23}s_{24} + s_{12}^2s_{24}^2 + 2s_{12}^2s_{23}s_{34} + 2s_{12}s_{13}s_{23}s_{34} + 2s_{12}s_{23}^2s_{34} \\ &+ 2s_{12}^2s_{24}s_{34} + s_{12}s_{13}s_{24}s_{34} + 2s_{12}s_{23}s_{24}s_{34} + s_{12}^2s_{34}^2 + 2s_{12}s_{13}s_{34}^2 \\ &+ s_{13}^2s_{34}^2 + 2s_{12}s_{23}s_{34}^2 + 2s_{13}s_{23}s_{34}^2 + s_{23}^2s_{34}^2 \end{aligned} \tag{80}$$

$$\begin{aligned} (L_4)_{23}{}^{32} = -s_{13}s_{24}(3s_{12}s_{23} &+ s_{13}s_{23} + s_{23}^2 + 2s_{12}s_{24} + s_{13}s_{24} + s_{23}s_{24} \\ &+ 3s_{12}s_{34} + 2s_{13}s_{34} + 3s_{23}s_{34}) \end{aligned} \tag{81}$$

and $(L_4)_{32}{}^{32} = (L_4)_{23}{}^{23}\big|_{2\leftrightarrow 3}$ as well as $(L_4)_{32}{}^{23} = (L_4)_{23}{}^{32}\big|_{2\leftrightarrow 3}$. The q-expansion of its coefficient does not have any zero mode, consistent with the fact that the q^0 order of eq. (79) has to match the α'-derivative of the tree-level amplitude [37].

Cylinder diagrams as drawn in figure 1 can be interpreted not only as a one-loop process involving open strings but also as a tree-level exchange of closed strings [16]. In particular, the non-planar cylinder diagram gives rise to a propagator $\sim s_{12}^{-1}$ of gravitational states upon integration over q. Accordingly, the low-energy limit of double-trace open-string amplitudes at one loop reproduces the corresponding double-trace amplitudes in Einstein–Yang–Mills field theory [45]

$$A^{\text{tree}}_{\text{EYM}}(1,2,3|4,5) = s_{24}A^{\text{tree}}_{\text{SYM}}(1,3,2,4,5) - s_{34}A^{\text{tree}}_{\text{SYM}}(1,3,2,4,5)\,. \tag{82}$$

Indeed, the α'-expansions of the non-planar integrals $H^{12}_{123|45}$ & $\widehat{H}^{14}_{123|45}$ give rise to

$$\begin{aligned} A_{\text{cyl}}(1,2,3|4,5) = -\frac{1}{2}\int_0^1 \frac{\mathrm{d}q}{q}\, q^{\frac{s_{45}}{4}} \Big\{ & s_{45}A^{\text{tree}}_{\text{EYM}}(1,2,3|4,5) \\ + \; & \Big(\frac{\zeta_2}{2} - 12\mathscr{E}_0(4,0;q)\Big) s_{45}^3 A^{\text{tree}}_{\text{EYM}}(1,2,3|4,5) \\ + \; & 12\mathscr{E}_0(4,0;q)\,\big[s_{34}(s_{12}s_{23}s_{45}+2s_{12}s_{24}s_{45}+s_{45}s_{34}^2+s_{45}^2s_{34}+3s_{12}s_{24}s_{15}) \end{aligned} \tag{83}$$

$$\times A^{\rm tree}_{\rm SYM}(1,2,3,4,5) - (2 \leftrightarrow 3) \Big] + \mathscr{O}(\alpha'^5) \Big\}$$

and match the desired Einstein–Yang–Mills limit eq. (82) by means of the integral $\int_0^1 \mathrm{d}q\, q^{\frac{s_{45}}{4}-1} = \frac{4}{s_{45}}$ at the leading order. It would be interesting to explore the higher-order structure of the α'-expansion at one loop, in particular, if it exhibits an echo of the tree-level pattern of refs. [43, 46] under the motivic coaction.

6 Summary

In these proceedings, we investigate the appearance of eMZVs in one-loop amplitudes of the open superstring. In reviewing earlier results on the planar [4] and non-planar cylinder diagram [6], we streamline intermediate steps of the computations provided in the references, thus allowing a more efficient calculation. We extend their results in two directions: First, the treatment of kinematic poles in planar and non-planar five-point integrals is carefully explained. Second, the final expressions for the low-energy expansions at four and five points are cast into the language of iterated Eisenstein integrals.

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