UPPSALA UNIVERSITET

Intitutionen för Fysik och Astronomi

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Analytical Mechanics (1FA163)
Hjälpmedel: Physics Handbook, Mathematics Handbook

The exam consists of 5 problems for a total of 25 points. To pass, 13 points are required. Please answer each problem on a separate page (try avoid writing in the margins). Make sure your test code is on every page. Complete solutions, with motivations and figures, when appropriate, are required to get full points. **Good Luck!**

Useful formulae

- Hamiltonian: $H = p\dot{q} - L$, conjugate momenta $p = \frac{\partial L}{\partial \dot{q}}$
- Canonical equations: $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$
- Generating functions of canonical transformation $Q = Q(q, p, t)$, $P = P(q, p, t)$

$F_1(q, Q, t)$	$p = \frac{\partial F_1}{\partial q}$	$P = -\frac{\partial F_1}{\partial Q}$
$F_2(q, P, t)$	$p = \frac{\partial F_2}{\partial q}$	$Q = \frac{\partial F_2}{\partial P}$
$F_3(p, Q, t)$	$q = -\frac{\partial F_3}{\partial p}$	$P = -\frac{\partial F_3}{\partial Q}$
$F_4(p, P, t)$	$q = -\frac{\partial F_4}{\partial p}$	$Q = \frac{\partial F_4}{\partial P}$

- Poisson bracket: $\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$
- Hamilton-Jacobi equation: $H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$
- Action variable: $J = \oint p dq$, angle variable: $\varphi = \nu t + \delta$
- Time-dependent perturbation theory: $\dot{\alpha}_1 = -\frac{\partial \Delta H}{\partial \beta}\Big|_{\alpha_0,\beta_0}$, $\dot{\beta}_1 = \frac{\partial \Delta H}{\partial \alpha}\Big|_{\alpha_0,\beta_0}$
- Time-independent perturbation theory $\alpha_1(J) = \overline{H}_1$, $\nu_1 = \frac{\partial \alpha_1}{\partial J}$

1. True/False (no motivation needed):

a) The variational principle for deriving the canonical equations requires paths that are fixed at the endpoints in phase space.

b) Sufficiently near any extremum, the motion in a potential is described by harmonic oscillator motion.

c) If the Hamiltonian is independent of time then it equals the total energy of the system.

d) Suppose that the Hamiltonian $H(q,p)$ and the function $A(q,p,t)$ are both constants of motion. Then $\{H,A\} = 0$.

e) Let $f = f(q_i + p_i, t)$ and $g = g(q_i - p_i, t)$ be some functions. Then $\{f,g\} = 0$.

f) If phase space is $2n$-dimensional then the maximum number of mutually Poisson-commuting variables is n.

g) Action–angle coordinates can always be defined for a Hamiltonian $H(p,q)$ provided that the range of both p and q are finite.

h) Since the Kepler problem is integrable the Lyapunov exponent is zero for all trajectories.

i) Assuming action–angle coordinates, the relations between the frequencies has to be irrational for the system to be well behaved under small perturbations.

j) In time-dependent perturbation theory the α, β variables become adiabatic invariants.

2. Short Answers:

a) Briefly describe different approaches for obtaining, or showing that, a phase-space coordinate transformation $Q_i = Q_i(q,p,t)$, $P_i = P_i(q,p,t)$ preserves the form of the canonical equations. Give at least three different approaches. (2pts)

b) Consider the linearized equations of motion near a fixed point, and perform a canonical transformation such that the canonical equations simplify to

$$\dot{Q} = \lambda_1 Q, \quad \dot{P} = \lambda_2 P$$

Discuss if this simple form is possible for general (linearized) systems and fixed points. Also describe the properties of λ_1, λ_2 for different types of fixed points. (2pts)

c) Briefly explain Noether's theorem and give an explicit example where it applies. (1pt)

3. Using Cartesian coordinates in the plane, the Lagrangian for the motion of the Foucault pendulum may be written as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\Omega^2(x^2 + y^2) + \omega_z(x\dot{y} - y\dot{x})$$

where $\Omega = \sqrt{\frac{g}{\ell}}$ is the usual pendulum frequency and ω_z represents the Coriolis force. In the (x, y) coordinates, the motion is inseparable.

a) Write down the Hamiltonian in polar coordinates in the (ρ, ϕ) plane. Show that the Hamiltonian becomes separable in these coordinates. (1.5pts)

b) Solve the Hamilton-Jacobi equation, i.e., find Hamilton's principal function for this problem up to elementary integrals. (1.5pt)

c) Find the action variables J_ρ, J_ϕ corresponding to the variable pairs (ρ, p_ρ) and (ϕ, p_ϕ), respectively. Choose a value for p_ϕ that allows you to evaluate the J_ρ integral, and describe in words the physical situation that corresponds to this choice, i.e. how does the trajectory look like in the (x, y) plane? (2pt)

4. The Hamiltonian of a one-dimensional dynamical system is

$$H(q, p, t) = 2e^{2t}(1 + pq^2)$$

a) Show that

$$A = (1 + ae^{2t}q + be^{-t}p)(1 + pq^2)\frac{1}{q}$$

is a conserved quantity for some choice of the parameters a and b. (1pt)

b) Using the Hamilton-Jacobi method find the principal function S for this system. (1pt)

c) Express the conserved quantity A in terms of the new variables obtained from the canonical transformation generated by S. (1pt)

d) Using the above results identify a second constant of motion (expressed in term of the original variables). (1pt)

e) Give the complete solution to the system in the original variables. (1pt)

5. A particle of mass m moves in one dimension under the infuence of a potential

$$V(x) = F\,|x|$$

where F is a constant.

a) Sketch the potential and carefully determine the conditions under which action-angle variables can be used. (0.5pts)

b) Calculate the action-angle variables for a trajectory of energy E. What is the period of motion T ? (1pt)

c) Suppose the parameter F is varying adiabatically (slowly changing). What happens to the energy of the particle? The period? The amplitude of oscillation? (1.5pt)

d) Let $F \to F + \epsilon\tilde{F}$, where $\epsilon \ll 1$ and $\tilde{F}$ can be assumed to be constant over one period. Use time-independent perturbation theory to determine the shift in energy and period to first order in ϵ. Compare the answer to that of part c) (2pts)